Arithmetic Revisited

Lesson 5:

Decimal Fractions or Place Value Extended

Part 7: An Introduction to Irrational Numbers

1. Prelude

It is not uncommon for a person who is not musically gifted to take a course called "Music Appreciation". Nor does a person have to be an artist to enjoy an "Art Appreciation" course. In this context, there are aspects of mathematics that have an esthetic value that is important quite apart from any practical value. There are mathematicians who study mathematics for the same reason that humanities majors study poetry - because it is beautiful!

Even some of the most profound practical mathematical observations involve an imagination that almost defies description. In fact we have already seen one aspect of this when we discussed Avogadro's Number (Lesson1 Part 3)[1].

> **If we were to try to count this number (a 6 followed by 23 zeroes) and we could count at a rate of 1 billion atoms per second continuously, it would still take over 18 million years to count that many atoms.**

Can you name any humanist who had a more vivid imagination? Yet much of our modern theory of the molecular structure of gases rests on the "vivid imagination" of Avogadro!

While it might not seem nearly as spectacular, our discussion in the previous part of this lesson leads us to another result that runs counter to the intuition of most of us; namely that there are numbers that are not rational. That is, there are numbers that cannot be expressed as the quotient of two whole numbers. Such numbers are called irrational numbers and they will be discussed in this part of our lesson.

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[1]Review: Amedeo Avogadro (1776-1856) the Italian chemist and physicist advanced a hypothesis that has come to be called Avogadro's Law.. Avogadro's law states that equal volumes of all gases under identical conditions of pressure and temperature contain the same number of molecules. From this hypothesis other physicists were able to calculate that there were approximately 6×10^{23} atoms in a mole of any substance.

2. An Introduction to Irrational Numbers

For a long time it was assumed that any number could be expressed as the quotient of two whole numbers. In other words, it was assumed that all numbers were rational numbers.

> **However with what we have just demonstrated, it becomes clear that there are numbers that are not rational.**

In particular, we showed that in terms of its decimal representation, a rational number either terminated or else eventually repeated the same cycle of digits endlessly.

-- Therefore any non-terminating decimal that doesn't eventually repeat the same cycle of digits endlessly cannot represent a rational number.

-- For example, take a sequence such as 28 and each time add another 8 to the cycle; that is, 0.28288288828888....(where "..." means you're adding another 8 to each cycle). The resulting decimal is non-terminating but also non-repeating (because each time there is an extra 8 per cycle). Hence it doesn't represent a rational number.

-- Here's an interesting way to help appreciate the vastness of the irrational numbers. Imagine that there is a spinner (a "wheel of fortune" so to speak) that has the digits 0 through 9 on it. Construct a decimal as follows.

- **Write the decimal point and then spin the wheel.**
- **Whatever number the spinner stopped at, record that as the first decimal digit (that is, the tenths digit).**
- **Spin the wheel a second time, this time recording the digit at which it stopped as the second decimal digit (that is, the hundredths digit)**
- **Keep repeating this process, each time recording as the next decimal place the digit at which the spinner stopped.**
- **Assume this process is continued "endlessly".**

How likely do you think it is that if you were to follow the above process that you would eventually get the *same cycle of digits* to *repeat <u>endlessly</u>*? In fact, in terms of the process described above it seems almost impossible that you'd ever get the same cycle to repeat endlessly! Yet ths happens "a lot" in the sense that there are infinitely many ratonal numbers!

Hopefully this helps you visualize why even though there are an infinite number of rational numbers, this "infinity" is actually smaller than the "infinity" of the irrational numbers.

3. A Note About "Endlessness"

Suppose there are a group of people in a room and all of them decide to change their names. Does this have any bearing on how many people are in the room?

-- In fact to make this problem seem more mathematical, suppose there were ten people in the room and we numbered them from 1 through 10. Suppose we then ask each person to change his name to whatever was twice the number he now had.

-- For example, #1 would now become #2; #2 would now become #4 and so on. In this way there would still be ten people but instead of having the names 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10; they would now have the names 2, 4, 6, 8, 10, 12, 14, 16, 18 and 20.

-- The same thing would happen even if there were a billion people. That is, if the people were numbered from 1 through 1,000,000,000 and each changed their name to twice the number they originally had; there would still be a billion people but they would now be named by the even numbers from 1 through 2,000,000,000.

Now try to imagine that there are an "endless" number of people in the room and that we were able to number them. Their names would now be 1, 2, 3, 4, 5, 6, etc. but now the list would never end! In fact the list of their names would be synonymous with the set of all the non-zero whole numbers.

-- However, now suppose they all change their name to twice the number they originally had (that is, 1 becomes 2, 2 becomes 4, 3 becomes 6, 4 becomes 8, etc.). In this case all the odd numbers have disappeared. That is:

Original Name →	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
New Name →	2	4	6	8	10	12	14	16	18	20	22	24	26	28

-- There is a "cute" way to restate the above remark. Imagine a hotel that has infinitely many rooms numbered from 1 to "infinity" and that all the rooms are filled. The room clerk, to make more room, asks each guest to move to the room that is twice the number of his or her current room. *In this way the even numbered rooms are still completely occupied; but every one of the odd numbered rooms is now vacant*!

The point of the above discussion is to prepare you for the fact that ideas that are quite "simple" is finite situations are mind boggling when we try to apply these ideas to infinite (endless) situations. For example, suppose

you start with the decimal 0.33 and delete one of the 3's to get a new decimal 0.3. Clearly 0.3 and 0.33 are not equal (in fact 0.33 exceeds 0.3 by 0.03).

Suppose next that we started with 0.3333333333 and deleted one of the 3's to get the new decimal 0.333333333. Again we get two different decimals even though their difference, 0.0000000003, is quite small.

Even if there were a billion (or a trillion or any *finite* but arbitrarily large number) of 3's after the decimal point, deleting one of the 3's would make the new decimal less than the original decimal.

However, if there are an endless number of 3's after the decimal point and you delete one of them (or any *finite* number of them), there are still an endless number of 3's after the decimal point. Most people use the term "endlessly" as a form of exaggeration. For example a person might say that a job was "endless" instead of saying that the job took a very long time.[2]

In terms of what we are discussing in this module, here is a very interesting result; a result that to many may seem paradoxical or even incorrect. We will state it in the form of another illustrative example. Namely:

Illustrative Example:

Which inumber is greater, $0.\overline{9}$ or 1?

To grasp the point of this example, let's review the procedure we used to show that $0.\overline{3} = \frac{1}{3}$. That is:

$$\begin{array}{rcl} 10n & = & 3.\overline{3} \\ -\ 1n & = & 0.\overline{3} \\ \hline 9n & = & 3.0 \end{array}$$

Now let's use the same technique to express $0.\overline{9}$ as an equivalent common fraction. Letting $p = 0.\overline{9}$, we see that:

$$\begin{array}{rcl} 10n & = & 9.\overline{9} \\ -\ 1n & = & 0.\overline{9} \\ \hline 9n & = & 9.0 \end{array} \quad \textbf{... and hence } n = 1$$

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[2]One way to see the difference between " a lot" and endless is to let N stand for the "greatest number you can imagine". For example,consider the number which in place value is represented by a 1 followed by 10 billion zeroes. In exponential notation this number is $10^{10,000,000,000}$ (and in more compact notation it can be represented as $10^{(10^{10})}$. Let N denote this number. Since a billion seconds is in excess of 30 years, at the rate of 1 digit per second, it would take over 300 years just to write this number using place value notation! However, if we start counting after N, we get $N+1$, $N+2$, $N+3$, etc. and we are once again back to the beginning of our counting system only using N as if it were 0.

If we accept the fact that a number has one and only one value, the fact that n is equal to both 1 and $0.\overline{9}$ means that $0.\overline{9} = 1$. [3]

Yet how can this be? If we write any number of 9's after the decimal point, whenever we stop the resulting decimal will name a number that is less than 1. For example:

-- 0.9 is less than 1. In fact, $1 - 0.9 = 0.1$

-- 0.99 is less than 1. In fact, $1 - 0.99 = 0.01$

-- 0.999 is less than 1. In fact $1 - 0.999 = 0.001$

Do you see the pattern? Every time we annex one more 9 the resulting decimal is greater than the previous decimal; yet it's still less than 1! And even if we wrote a trillion 9's after the decimal point, *when we were <u>finished</u>* the decimal would still be less than 1.

> **The paradox here is the key phrase is "when we finish"; namely how does one *finish* an endless process?**

So, strange as it may seem, once we agree that the sequence of 9's never ends, the resulting decimal, $0.\overline{9}$, is equal to 1.

***<u>Note</u>*:**

Actually, what we can show is that it's impossible to find a number that's less than 1 but greater than $0.\overline{9}$? More specifically, recall that the way we compare the size of two decimals is to look at them place by place. We then look for the first place in which the two decimals are different; in which case the greater of the two digits names the greater decimal.

However, since it is impossible for a digit to be greater than 9, there is no number less than 1 that's greater than $0.\overline{9}$. In other words, *no number exists that is between $0.\overline{9}$ and 1*.

And if we make the rather "natural" assumption that the number line is continuous (that is, it has no "gaps") it must be true that $0.\overline{9} = 1$.

In summary, even if we have a trillion 9's following the decimal point, there is an even larger number that is still less than 1. But when it comes to "endlessness", even a trillion is but a drop in the bucket!

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[3] Another way to get this result is by starting with the fact that $\frac{1}{3} \times 3 = 1$; and then replacing $\frac{1}{3}$ by $0.\overline{3}$ to obtain $0.\overline{3} \times 3 = 1$. Since $0.\overline{3} \times 3 = 0.\overline{9}$, it again follows that $0.\overline{9} = 1$.

4. A Geometric Interpretation of "Endless" Versus "Very Much"

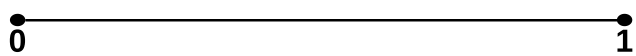
The Ancient Greek mathematician, Euclid, defined a point to be "that which has no parts". Now if something has "no parts" it must be invisible.

> **-- That is, if it were visible, no matter how tiny it was, it could still be divided into smaller "parts".**
>
> **-- For example, suppose the point was as small as " · ". The fact that it is now visible means that under a powerful enough microscope we could greatly enlarge it so that it might now look like •, in which case "part" of it would still be visible.**

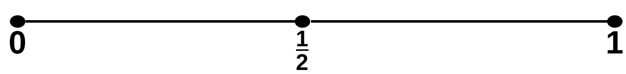
However, if it were invisible we couldn't see it.

> **Therefore even though we talk about points (no thickness) we represent them by "dots"(which do have thickness).**

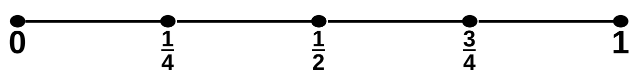
No matter how thin the dot is, it still isn't a point; but the difference is very subtle. For example, let's look at a line that is 1 unit in length.



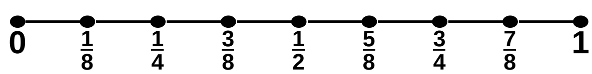
We have made the dots that represent the points 0 and 1 rather thick in order to emphasize their visibility. We then locate the point, $\frac{1}{2}$, that is midway between 0 and 1

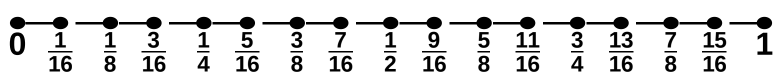


which divides our line segment into two pieces of equal length. We next divide each of these two pieces into two more pieces of equal length



We continue in this way, obtaining successively (not drawn to scale)





After a while the dots run together; that is: there is no longer any space between them.

Now the ancient Greeks were intelligent enough to recognize that the "dots" they had used were rather large. However, their argument was that no matter how much smaller the "dots" had been there would eventually have been no spaces between them. For example, suppose they had used a new "dot" that was so thin that it took a thousand of them to have the same width as one of the original dots. All that would have meant is that there would have been a thousand times as many new "dots" as there were old "dots" between 0 and 1. It would still be only a finite number of "dots".

With this type of reasonable logic it was easy to see why the ancient Greeks believed that the points (which were represented by the "dots") would eventually fill in the entire number line; and that, therefore, all numbers were rational numbers. However, notice that the procedure of each time picking a new "dot" to be midway between two consecutive old "dots" gives us only those common fractions whose denominators were powers of two. In other words, no matter how thin the "dots" were; when the line was filled in, most of the common fractions weren't even included.

What happened, of course, is that after a while the "space" between say $\frac{1}{3}$ and a "dot" that represented a fraction whose denominator was a power of 2 became smaller than the thickness of the "dots" that were being used.

-- For example, the first "few" powers of 2 are 2, 4, 8, 16, 32, 64 and 128. Suppose that the thickness of each "dot" was $\frac{1}{50}$ of an inch.

-- The least common multiple of 64 and 3 is 192. In terms of 192 as a denominator, $\frac{1}{3}$ is represented by $\frac{64}{192}$ which lies between $\frac{63}{192}$ (which reduces to $\frac{21}{64}$)and $\frac{63}{192}$ (which reduces to $\frac{22}{64}$).

-- Thus $\frac{1}{3}$ would be "trapped" between $\frac{21}{64}$and $\frac{22}{64}$; a "distance" of less than $\frac{1}{50}$ of an inch.

-- In other words, the thickness of the "dots", no matter how thin they were, would eventually make it look as if $\frac{1}{3}$ was represented by one of the "dots".

5. Some Concluding Remarks:

The study of irrational numbers is a wonderful example of how a topic that has no practical value (at least to most people) can occupy the minds of even the greatest mathematicians; and at the same time can arouse the curiosity of most lay persons.

For example, in our discussion of "filling in the number line" with rational numbers, do you think that there are many folks who would deny that it was virtually self-evident that the rational numbers did, indeed, fill in the entire line? Yet as self-evident as this might have seemed it is not true. Yet the truth or falsity of this "self-evident" observation is irrelevant with respect to the real-world. Namely with a number such as 0.282882888.... (where each time one more 8 is added to the cycle), we would use a rational number approximation (such as 0.282888) if we needed to use this number in a real-life application. Indeed this happens when we use the irrational number π in a mathematical computation.[4] To 6 decimal place accuracy, the value of π is 3.141592. Yet, depending on the degree of accuracy we need, we approximate π by such rational numbers as 3.14, 3.1416 and $\frac{22}{7}$, etc.

Notice, for example, that while $\pi = 3.141592...$, we might round it off to 3.14 or possibly $\frac{22}{7}$ in computing the area or the circumference of a circle whose radius is known. For example if we are told that the radius of the circle, measured to 2 decimal place accuracy is 6 inches, there is no need to know the value of π beyond the second decimal digit. In decimal form $\frac{22}{7}$ is $3.\overline{142857}$; and rounded off to 2 decimal place accuracy, both π and $\frac{22}{7}$ become 3.14. However, if the radius had been measured even more accurately, the approximation 3.14 might no longer be adequate.

As another example that is not self-evident, it turns out that there is no rational number which when multiplied by itself is equal to 2. In more mathematical terms this means that $\sqrt{2}$ is an irrational number. However we can estimate the value of $\sqrt{2}$ to as many decimal place accuracy that we may need.

[4] π (the Greek letter "pi") is best known as being the ratio between the circumference and the radius of a circle. Since all circles have the same shape, the value of π is independent of the circle we choose.

For example:

$$1.4^2 = 1.96 \text{ and } 1.5^2 = 2.25$$

Hence $\sqrt{2}$ is greater than 1.4 but less than 1.5.

$$1.41^2 = 1.9881 \text{ and } 1.42^2 = 2.0164$$

Hence $\sqrt{2}$ is greater than 1.41 but less than 1.42[5]

Yet even though we can always replace an irrational number by a sufficiently accurate rational number approximation, a great deal of mathematics is devoted to the study of irrational numbers.

The late Samuel Eilenberg made a poignant statement when asked how he juxtaposed his life as a "pure" mathematician with his life as a great "applied" mathematician. His eloquent response was words to this effect: "I compare myself to an esthetic tailor, making jackets as the mood strikes me. Sometimes I make them with 32 arms, 55 legs, and 25 different colors. Every now and then, however, I make a conventional coat with two sleeves, no legs and one color. If a fellow human being comes into my shop and likes the jacket, I give it to him as a favor out of the goodness of my heart."

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[5]Notice that we often do such things even when dealing with rational numbers. Fir example in doing a calculation on a calculator we might round off $\frac{1}{3}$ to 0.3, 0.33, 0.333 etc, depending on the desired degree of accuracy.