Arithmetic Revisited

Lesson 5:

Decimal Fractions or Place Value Extended

Part 5: Dividing Decimal Fractions, Part 2

1. The "Fly In The Ointment"

The meaning of, say, $1 \div 2$ doesn't depend on whether we represent the quotient as a common fraction or as a decimal fraction. More specifically, independently of how we elect to write the answer, $1 \div 2$ means the number that when multiplied by 2 yields 1 as the product. However if we elect to write the quotient as a common fraction we use the notation $\frac{1}{2}$; while in the language of decimal fractions, the quotient is represented by 0.5.

One possible disadvantage of using common fractions rather than decimal fractions is that there are many different common fractions that name the same rational number. For example, while $1 \div 2 = \frac{1}{2}$, it is also equal to $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, etc. On the other hand, when we use decimal fractions there is only one way to represent the quotient $1 \div 2$; namely, as 0.5.[1]

On the other hand, using common fractions allows us to express the quotient of any two whole numbers rather quickly. For example, in the language of common fractions, $139 \div 201 = \frac{139}{201}$.[2]

However, representing this quotient in the form of a decimal fractions presents a new challenge to us.

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[1]Actually, there is a fine point that we are ignoring. Namely while $1 \div 2 = 0.5$, it is also equal to 0.50, 0.500, etc. However to avoid this technicality we will agree to end the decimal representation after the last non-zero digit. For example, we will write 0.25 rather than 0.25000, etc. This is analogous to what we do when we use common fractions. Namely we pick the fraction which has been reduced to lowest terms. For example, even though $\frac{1}{2} = \frac{2}{4}$, we do not usually write that $1 \div 2 = \frac{2}{4}$.

[2]More generally, if m and n are whole numbers and $n \neq 0$ then $m \div n = \frac{m}{n}$.

To see why let's see what happens when we try to represent the quotient $5 \div 6$ as a decimal fraction. Clearly it is a simple task to write the quotient as a common fraction, namely: $5 \div 6 = \frac{5}{6}$. Proceeding as we did in the previous part of this lesson we can use the division algorithm, to obtain:

$$\begin{array}{r|l} & 0.\quad 8\quad 3\quad 3\quad 3\quad 3 \\ \hline 6 & 5.\quad {}^{5}0\quad {}^{2}0\quad {}^{2}0\quad {}^{2}0\quad {}^{2}0 \end{array}$$

and no matter how many 0's we use to augment the dividend, we are faced with the fact that each time the algorithm tells us that "6 goes into 20 three times with a remainder of 2". In other words, the decimal fraction that represents $5 \div 6$ consists of a decimal point followed by an 8 and an *endless* number of 3's!

This is where the fly gets into the ointment. In this particular example, we saw that a problem arises when we try to express the common fraction $\frac{5}{6}$ as an equivalent decimal fraction. Unfortunately, this situation turns out to be the general rule rather than the exception. More specifically:

> **Given two whole umbers chosen at random, it is highly likely that the decimal fraction that represents their quotient will be an "endless" decimal.**

Why this is the case is the topic of the next section.

2. When Will the Quotient Be a Terminating Decimal?

For the common fraction to represent a decimal that terminates, its denominator must be a power of 10. That is, a terminating decimal must be a whole number of tenths, hundredths, thousandths etc. However, the only prime factors of powers of 10 are 2 and 5. Hence if a divisor *i.e., the denominator) has even one prime factor other than 2's and 5's, the decimal representation will never come to an end.

<u>Illustrative Example</u>:

> **Express $\frac{3}{8}$ as an equivalent common fraction whose denominator is a power of 10.**

We may "decompose" 8 into a product of prime numbers by writing it as $2 \times 2 \times 2$. If we multiply 2 by 5 we get 10. Hence if we multiply each of the three 2's by 5 we get 10^3 (or 1,000) which is, of course, a power of 10. More specifically:

$$2 \times 2 \times 2 \times 5 \times 5 \times 5 =$$
$$(2 \times 5) \times (2 \times 5) \times (2 \times 5) = 10 \times 10 \times 10 = 10^3$$

However, if we multiply the denominator of a fraction by $5 \times 5 \times 5$, we must also multiply the numerator by $5 \times 5 \times 5$; otherwise we change the value of the fraction.

Hence we may rewrite $\frac{3}{8}$ in the form $\frac{3\times5\times5\times5}{2\times2\times2\times5\times5\times5}$ or $\frac{375}{1{,}000}$.

As a decimal $\frac{375}{1{,}000}$ is 0.375

***<u>Commentary</u>*:**

- **If we wanted to do this problem strictly by using decimals, we know that $\frac{3}{8}$ is the answer to $3 \div 8$. If we now proceed as we did earlier, we see that:**

	0.	3	7	5
8	3.	0	0	0
−	2	4		
		6	0	
	− −	5	6	
			4	0
		− −	4	0
				0

- **Notice that in doing the division we finally arrived at a point where the reminder was 0 when we subtracted.**

Practice Problem #1

Express the common fraction $\frac{3}{40}$ as an equivalent decimal fraction.

Answer: 0.075

Solution:

We observe that the only prime factors of 40 are 2 and 5. More specifically:

$$40 = 2 \times 2 \times 2 \times 5.$$

To obtain a factor of 10 we must multiply 2 by 5. Since 40 has three factors of 2 but only one factor of 5, we need two more factors of 5 in order to obtain a power of 10.

With this in mind we multiply both numerator and denominator of $\frac{3}{40}$ by 5×5 to obtain:

$$\frac{3}{40} = \frac{3}{2\times2\times2\times5} = \frac{3\times5\times5}{2\times2\times2\times5\times5\times5} = \frac{3\times5\times5}{10\times10\times10} = \frac{75}{1{,}000} = 0.075$$

Notes:

- **We could again have obtained the same result by long division. Namely:**

$$\begin{array}{rrrrrr}
 & & 0. & 0 & 7 & 5 \\ \cline{3-6}
40 & \big) & 3. & 0 & 0 & 0 \\
 & - & 2 & 8 & 0 & \\ \cline{3-5}
 & & & 2 & 0 & 0 \\
 & & - & 2 & 0 & 0 \\ \cline{4-5}
 & & & & & 0
\end{array}$$

- **Once again notice that we finally arrived at a point where the remainder was 0 when we subtracted.**
- **We might also have observed that $3{,}000 \div 40 = 75$. Hence**

$3{,}000$ thousandths $\div\ 40 = 75$ thousandths

In the language of decimal fractions the above equality becomes

$$3{,}000 \div 40 = 0.075$$

Summary, So Far:

***Definition*:**

A decimal is said to *terminate* if it has a *last non-zero digit*.

-- Another way of saying this is to say that beyond a certain number of places, the decimal has nothing but 0's.

• What we have shown in this problem is that the fraction $\frac{3}{8}$ can be represented by the *terminating* decimal 0.375 (as well as 0.3750, 0.37500, etc.).

• The important thing is that in order for a fraction to be represented by a *terminating* decimal, *it must be equivalent to a common fraction whose denominator is a power of 10*.

• However, the *only prime numbers* that are factors of 10 are *2 and 5*.

• Hence, if when written in lowest terms, a common fraction has a prime factor *other than 2 and/or 5*, it will *never* be equivalent to a fraction whose denominator is a power of 10.

One might think that since we can write as many 0's as we wish after the decimal point that we are eventually bound to come to a place where the decimal will terminate. Unfortunately, as we mentioned earlier, this will happen only if the denominator has no prime factors other than 2 and/or 5.

To get an idea of what happens if the denominator does contain a prime factor other than 2 or 5, look at the following example.

Illustrative Example

Is there a common fraction whose denominator is a power of 10 that is equivalent to the common fraction $\frac{1}{3}$?

The key observation for answering this question is to recall that a number is divisible by 3 if and only if the sum of it's digits is divisible by 3. Since any power of 10 has as its digits a 1 followed only by 0's, the sum of its digits will always be 1 and hence never divisible by 3.

Commentary:

• When we talk about common fractions we will always assume (unless specifically stated to the contrary) that they are expressed in lowest terms. For example, the denominator of $\frac{3}{6}$ is 6; which has a prime factor other than 2 or 5. However, if we reduce $\frac{3}{6}$ to lowest terms it becomes $\frac{1}{2}$, which is represented by the terminating decimal 0.5.

• The result has an interesting interpretation if we think in terms of decimal fractions. Namely:

	0.	3	3	3
3	1.	0	0	0
	– –	9		
		1	0	
	–	–	9	
			1	0
		–	–	9
				1

The above process shows us that at each step in the division process, we are saying "3 goes into 10 three times with a remainder of 1". Thus no matter how many places to the right we extended our quotient, there would always be a remainder of 1; and, hence, never a remainder of 0.

The fact that there are many more numbers whose prime factorization contains prime numbers other than 2's and or 5's means that when we express rational numbers as decimal fractions, the decimals will most likely be non-terminating.[3]

<u>Practice Problem #2</u>

Will the decimal fraction that is equivalent to the common fraction $\frac{9}{15}$ terminate?

Answer: Yes

[3] For example, if we look at the whole numbers that are greater than 1 but less than 100, we see that only 2, 4, 5, 10, 16, 20, 25, 32, 40, 50 and 80 have no other prime factors other than just 2's and/or 5's.

Solution:

This can be a bit tricky. Namely, the denominator has a factor other than 2 or 5. however the fraction is not in lowest terms. In fact, we can cancel the common factor 3 from the numerator and denominator and see that:

$$\frac{9}{15} = \frac{3}{5} = \frac{3\times2}{5\times2} = \frac{6}{10} = 0.6$$

Note:

- **We could have obtained the same result by the division algorithm. Namely:**

$$\begin{array}{r|l} & 0.\ 6 \\ \hline 15 & 9.\ 0 \\ - - & 9\ \ 0 \\ \hline & \ \ \ \ 0 \end{array}$$

However, we wanted to emphasize the role of the denominator in converting a common fraction into a decimal fraction.

Practice Problem #3

Will the decimal fraction that is equivalent to the common fraction $\frac{2}{15}$ terminate?

Answer: No

Solution:

This time the fraction is in lowest terms. Therefore, since any power of 10 has only 2's and/or 5's as prime factors, 15, which contains 3 as a prime factor can never be a divisor of any power of 10.

Note:

Once again we could have gone directly to the division algorithm and obtained:

```
           0.  1   3   3
     15  | 2.  0   0   0
    – –    1   5
           ---------
               5   0
           –   4   4
               ---------
                   5   0
               –   4   5
                   -----
                       5
```

It soon becomes clear that we can go on "forever" saying "15 goes into 50 three times with a remainder of 5". This leads in a natural way into our next section.

3. An Interesting Property of Decimal Fractions.

When we tried to write $\frac{1}{3}$ as a decimal fraction we saw that the division yielded an endless sequence of 3's. When we tried to write $\frac{2}{15}$ as a decimal fraction we saw that after the digit in the tenths place (namely, 1) we again obtained an endless sequence of 3's. ***The fact is that when any rational number is written in decimal form, the decimal will either terminate or else it will eventually repeat the same cycle of digits endlessly.*[4]** **As an example (but not a proof of our assertion), let's try to represent $\frac{6}{7}$ as a decimal fraction. Using the "short division" format we see that**

```
      0.  8   5   7   1   4   2          . . . .
  7 | 6.  0  ⁴0  ⁵0  ¹0  ³0  ¹0  ⁶0   0
```

Notice that when we started the division process our dividend was a 6 followed by nothing but 0's; and after the sixth decimal digit our dividend was again a 6 followed by nothing but 0's. Hence the cycle of digits "857142" will repeat endlessly.

[4]In fact if we think of a terminating decimals as eventually repeating 0's endlessly we may say that all rational numbers can be expressed as non-terminating, repeating decimals.

Was it a coincidence that in the two examples we picked in which the decimal fraction did not terminate the decimal eventually repeated the same cycle of digits endlessly? The answer is that it wasn't a coincidence. The proof is actually quite elementary. It is based on what is known as *the Dedekind "Chest-of-Drawers" Principle*. The example below gives a specific application of the principle (and why it has the name that it does!)

<u>Illustrative Example</u>

***Imagine that in a bureau drawer there are 100 separate white socks and 100 separate black socks. The room is dark and you want to make sure that you pick a matched pair of socks (that is, either 2 white socks or 2 black socks). What is the least number of socks that you can take out of the drawer and still be sure that you have such a pair*?**

If all you took were 2 socks you might have a matched pair but you may also have picked one white and one black sock from the drawer. It's possible that the first two socks you picked gave you a matched pair. However if this wasn't the case, it means that you have chosen one white and one black sock. Therefore, since the next sock you choose must be either while or black, you will then have either a black pair or a white pair. The point is that since there are only two colors, you are guaranteed to have a mixed pair if you pick any three socks.[5]

As a second example, suppose there are 367 people in a room. Then even allowing for a Leap Year, at least two of the people in the room must have been born on the same day (but not necessarily in the same year). Namely, let's look at the case in which there are 366 people in the room. If we already can find 2 who do then we've proven our claim. So suppose no two of them celebrate their birthday on the same day. Since there are 366 people in the room, it means that between them they have used up every possible day. Therefore, in this case if another person enters the room, the birthday of this 367th person must be celebrated on the same day that one of the others is celebrating.

[5]Notice that we aren't saying what color the matched pair is. It would be a quite different if we had said "What is the least number of socks you can take from the drawer to make sure that you have a *white* pair?".

Practice Problem #4

Imagine that in a bureau drawer there are 100 separate white socks, 100 separate brown socks, 100 separate yellow socks and 100 separate black socks. The room is dark and you want to make sure that you pick a matched pair of socks. What is the least number of socks that you can take out of the drawer and still be sure that you have such a pair?

Answer: five

Solution*:*

There are only 4 different colors. So if you took out only 4 socks, there is a chance that by a stroke of either good or bad luck, you managed to get 4 different colored socks. However,even in this worst case scenario, no matter how lucky or unlucky you deem yourself to be, if you now take out a fifth sock, it has to match one of the other four socks.

These rather simple illustrations are a form of Dedekind's “Chest-of-Drawers” Principle. More specifically:

***Dedekind's “Chest of Drawers” Principle*:**

If you have more items than you have drawers to put them in, at least one of the drawers must hold more than 1 item (actually, all of the items could be in the same drawer; but at least two of them must be).

4. Dedekind's Principle and Non-Terminating Decimal Fractions

Let's now apply this idea to converting common fractions into decimals. In particular, let's revisit the case of $\frac{1}{3}$. When we divide a number by 3 there are 3 possible remainders; namely, 0, 1, or 2.[6] This concept applies to any denominator we may be using. For example, if there were 7 books in a carton, when all the books were packed, there could be at most 6 books left because if there were 7 we could have filled another carton. That is, when we divide a whole number by 7, there are 7 possible remainders; 0, 1, 2, 3, 4, 5, or 6.[7]

> **In general when we divide a number by any non-zero whole number n, there are n, possible remainders. Hence when we write the common fraction $\frac{m}{n}$ as a decimal fraction, the decimal part must begin to repeat no later than by the time we get past the nth place in the decimal.**

The fact that the decimal fraction that represents a rational number eventually repeats the same cycle of digits gives us a nice way to write such a decimal. Namely, we simply place a bar over the repeating cycle of digits. For example,

$0.\overline{3}$ stands for 0.3333......(where the dots indicate the decimal never ends)

$0.\overline{216}$ stands for 0.216216......

$0.2\overline{16}$ stands for 0.216161616.....

Non-terminating decimal fractions are an interesting intellectual topic but they are not necessary in the "real world". For example, even though we can't express $\frac{1}{3}$ exactly as a terminating decimal, we can use terminating decimals to get very good approximations. For example,

$$0.33 = \frac{33}{100} = \frac{99}{300} \text{ and } \frac{1}{3} = \frac{100}{300}.$$

Hence $\frac{1}{3} - 0.33 = \frac{100}{300} - \frac{99}{300} = \frac{1}{300}$

[6] An easy way to see this is to think of packing books into cartons, each of which holds 3 books. When all the books are packed, there can be either none, 1 or 2 books left because if there were at least 3 books left, we could have filled another carton.

[7] (Be careful here, there are not 6 remainders but 7. That is, there are 6 whole numbers between (and including) 1 and 6 but there are 7 whole number between 0 and 6.

-- Thus, for example, if our measuring instrument cannot measure to closer than the nearest hundredth of an inch, since the difference between $\frac{1}{3}$ and 0.33 is $\frac{1}{300}$ (that is, an error 1 part per 300) we may in this case use 0.33 as a sufficiently accurate approximation for $\frac{1}{3}$.

-- And if we had a more sensitive measuring instrument, say one that could measure to the nearest millionth of an inch, we could use 0.333333 as an approximation for $\frac{1}{3}$ because:

$$\frac{1}{3} - \frac{333{,}333}{1{,}000{,}000} =$$

$$\frac{1{,}000{,}000}{3{,}000{,}000} - \frac{999{,}999}{3{,}000{,}000} = \frac{1}{3{,}000{,}000}$$

and an error of 1 part per 3,000,000 is less of an error than 1 part per 1 million.

In other words, when a common fraction cannot be expressed exactly as a terminating decimal, we can "chop off" the decimal (the technical term is that we say are *truncating* the decimal) after a sufficient number of places and use this as an approximation for the exact answer. [8]

A Note About Using the "Bar"

There is a tendency for some people to write "...." to stand for "and so on". The trouble with this notation is that what "and so on" means can vary from person to person. One nice example is the following sequence of numbers.

31, 30, 31, 30, 31, 31, 30, 31, 30, 31, 31,......

Based on what rule I am using to generate the above sequence of numbers, do you think you can you guess the next number? If you think you can, be warned that the next number is neither 30 nor 31. More specifically, the above list represents the number of days in a month starting with March in

[8]In fact, we do this quite often in mathematics. For example, the number π (which is the ratio between the circumference and diameter of a circle) was known to the Ancient Greek mathematician and philosopher, Archimedes, to be more than $3\frac{10}{71}$ but less than $3\frac{1}{7}$. It's actual value in decimal form begins with 3.14159...... However, in many situations we use $3\frac{1}{7}$ as the value of π. As you can easily verify , the decimal form of $3\frac{1}{7}$ begins as 3.142.... Notice that to the nearest hundredth this is the same value of π when it's rounded off to the nearest hundredth. Hence, even though $\pi \neq 3\frac{1}{7}$, whenever we do not need more than accuracy to the nearest hundredth as may replace π by $3\frac{1}{7}$.

a non-Leap Year (in other words, the next number should be 28 because it is the number of days in February in a non-Leap Year).[9]

The point is that once we put the bar above the repeating cycle of digits, we no longer have to guess what "and so on" means. That is, if we see the notation $0.215\overline{67}$, we know at once that it means 0.215676767....; where in this case "and so on" means that the cycle "67" repeats endlessly..

<u>Practice Problem #5</u>

Write the first 7 digits of the decimal fraction .$0.\overline{847}$.

Answer: 0.8478478

<u>Solution</u>:

The bar over the sequence of digits 874 means that the cycle of digits "847" repeats endlessly after the decimal point. Hence the first 7 digits are 8478478.

<u>Notes</u>:

- **Again, the most common mistake students make is they simply ignore the bar and replace $0.\overline{847}$ by 0.847.**
- **0.8478478 is a very good approximation for $0.\overline{847}$. In fact:**

$$0.\overline{847} - 0.8478478 = 0.000000047\overline{847}.$$

- **In the event that the above error was deemed to be too great, we could approximate the value of $0.\overline{847}$ by writing more digits, for example, 0.8478478478 etc.**
- **In the real world we world eventually reach a point where we couldn't even measure the error because it would be too small.. In other words even if we didn't know the exact value of $0.\overline{847}$, we would eventually find a terminating decimal that was "close enough".**
- **With respect to this problem, rounded off to the nearest millionth, 0.8478478 becomes 0.847848; and what we could say in this case is that rounded off to the nearest millionth, $0.\overline{847} = 0.847848$**

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[9]Notice that if you had been told to write the number of days in each month of a non-Leap year starting with March" you would not have had any trouble guessing that the missing number was 28; but without your knowing what the pattern was, the phrase "and so on" could be easily misinterpreted by you. In particular, there are man ways to justify why you got a different answer but it still wouldn't have been the one that was required by the author of the sequence.

Practice Problem #6

Write the first 7 digits of the decimal fraction $0.8\overline{47}$

Answer: 0.8474747

Solution:

This resembles the previous problem except that the bar is only over the 4 and the 7. This means that the repeating cycle begins after the 8, Hence this time the decimal is 0.84747474747...; and rounded off to the nearest ten millionth it would be 0.8474747

Practice Problem #7

Is there a rational number that is denoted by $0.8\overline{7}4$?

Answer: No

Solution

This is analogous to asking "Is it possible to go 4 miles further after you've gone as far as is humanly possible?" In other words the bar over the 7 tells us that the 7 is repeated without end. Hence it cannot be followed by a 4.

5. A Closing Thought

There are many valid reasons to support why students prefer decimal fractions to common fractions. However there is one overriding advantage of common fractions that cannot be ignored. Namely much of our work with rational numbers is involved with the concept of rates; and finding a rate requires that we divide two numbers. Such a quotient is easily expressed in the language of common fractions. However, as we have just seen, problems can arise when we use decimal fractions to represent the quotient of two whole numbers.

Among other things, it is much easier to recognize the that $\frac{6}{7}$ represents the number that is defined by $6 \div 7$ than it is to recognize that $0.\overline{857142}$ is the answer to $6 \div 7$. Moreover, if wed did want to represent $6 \div 7$ as a terminating decimal, the best we could do is round our answer off to a certain number of decimal places.[10]

[10]For example, computing $6 \div 7$ on my calculator yields 0.857142857 as the answer, which we can round off to as many as 7 decimal place accuracy; but the resulting number, although adequate for most real-life applications, is just a good approximation to the exact answer...