****Unedited Draft****

**Arithmetic Revisited**

**Lesson 5:**

**Decimal Fractions or Place Value Extended**

**Part 1: Prelude**

## 1. Introduction: Review

**In our study of rational numbers we developed special fractions called *unit fractions*. For example, if we wanted to divide any number of pies equally among 3 people, we imagined that each pie was cut into three equally sized pieces and each piece was denoted by $\frac{1}{3}$. $\frac{1}{3}$ was read as "1 third" and meant that it was "1 of what it took 3 of to make a unit"**

**-- So, for example, if a particular measuring cup was labeled $\frac{1}{3}$, it meant that it was 1 of what it took 3 of to make 1 cup. A less cumbersome way to describe $\frac{1}{3}$ of a cup might be to say "3 of these equals 1 cup".**

**-- It was called a unit fraction for much the same reason that we think of 1 as a unit. Namely any whole number is a multiple of 1. In that sense, if any number of pies are being shared equally by 3 people, each person gets a multiple of $\frac{1}{3}$ of a pie. In other words, they get 1 third, 2 thirds, 3 thirds, 4 thirds, etc., depending on the number of pies.**

**-- In a similar way, for example, in dividing pies equally among 5 people, we would divide each pie into 5 equally sized pieces and call each of these pieces $\frac{1}{5}$. Thus the amount of pie each person got was a multiple of $\frac{1}{5}$.**

**More generally if we want to divide any number of pies equally among n people, we could divide each pie into n equally sized pieces and each piece would be denoted by $\frac{1}{n}$.[1] Then the amount of pie each person got would be a multiple of $\frac{1}{n}$.**

**The concept of decimals hinges on using only unit fractions whose denominators are powers of 10.**



[1]This is another way of seeing that $n \times \frac{1}{n} = 1$. Namely, each pie (that is, 1 pie) consists of $n$ pieces and the size of each piece is $\frac{1}{n}$.

## 2. A Special Set of Fractions

**While there are many physical models that can be represented by a pie (or corn bread"), it seems that money is something everyone understands. In this context**

**-- Because it takes 100 cents to equal 1 dollar, we may think of a cent as being the unit fraction $\$\frac{1}{100}$.**

**-- Because it takes 20 nickels to equal 1 dollar, we may think of a nickel as being the unit fraction $\$\frac{1}{20}$.**

**-- Because it takes 10 dimes to equal 1 dollar, we may think of a dime as being the unit fraction $\$\frac{1}{10}$.**

**-- Because it takes 4 quarters to equal 1 dollar, we may think of a quarter as being the unit fraction $\$\frac{1}{4}$.[2]**

**To capture the meaning of decimals, suppose that a nation wanted to invent a monetary system that was modeled in the image of place value. Every monetary system has to have a least denomination.[3] So let's suppose the nation decided to call the least denomination \$1[4]. Then the only other denominations would be the powers of 10. That is, they would have used only the denominations, \$1, \$10, \$100, \$1,000, etc.**

**Next suppose that they wanted to have denominations that were less than \$1 but that they still wanted to preserve the place value model. Since each denomination in place value is worth ten of the next smaller denomination, the first new denomination would have to satisfy the relationship**

$$10 \times \$\,? = \$1$$

**Because $10 \times \frac{1}{10} = 1$, we see that for the above relationship to be true, it must be that $\$? = \$\frac{1}{10}$.[5]**

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[2]We could have invented other denominations such as $\$\frac{1}{5}$ ("5 for a dollar", "2 dimes", "1 dime a nickel and 5 cents", etc.) and $\$\frac{1}{7}$ ("7 for a dollar" but no combination of existing coins equals $\$\frac{1}{7}$).

[3] In our own system that denomination would represent a cent; and for tax purposes we even use an old denomination known as a mill which is equal to $\frac{1}{10}$of a cent.

[4]Notice that the \$ being used here is not what we presently denote by \$. Namely it is naming the least denomination in our new monetary system.

[5]As we have already noted, in our own monetary system $\$\frac{1}{10}$ is called a dime. The word "dime" is derived from the word "decimal".

**And since $10 \times \frac{1}{10} = \frac{1}{10}$, we see that the next smaller denomination must be $\frac{1}{100}$.**

**In a similar way we see that while the whole number denominations are $1, $10, $100, $1,000 etc., the fractional denominations must be the reciprocals of these denominations; namely, $\$\frac{1}{10}$, $\$\frac{1}{100}$, $\$\frac{1}{1,000}$, $\$\frac{1}{10,000}$, etc.**

**<u>Practice Problem #1</u>**

**Rewrite $\frac{3}{5}$ as an equivalent common fraction whose denominator is 100.**

***<u>Answer</u>*: $\frac{60}{100}$**

**<u>Solution:</u>**

**5 is a divisor of 10. Hence we may rewrite $\frac{3}{5}$ as $\frac{3 \times 2}{5 \times 2}$ or $\frac{6}{10}$. We may then multiply numerator and denominator of $\frac{6}{10}$ by 10 to obtain $\frac{60}{100}$.**

**<u>Notes</u>:**

- **Since 5 is a divisor of every power of 10, we may also write such things as:**

$$\frac{3}{5} = \frac{6}{10} = \frac{60}{100} = \frac{600}{1,000} = \frac{6,000}{10,000}, \text{ etc.}$$

- **In terms of our corn bread model, decimal arithmetic is based on presliced corn breads, where the number of slices is a power of ten. In other words, $\frac{3}{5}$ of a corn bread is:**

  **-- 6 pieces if the corn bread consists of 10 pieces;**

  **-- 60 pieces if the corn bread consists of 100 pieces;**

  **-- 600 pieces if the corn bread consists of 1,000 pieces;**

  **-- 6,000 pieces if the corn bread consists of 10,000 pieces.** [6]

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[6] obviously we would most likely never divide a "real" corn bread" into 100,000 pieces. In other words, the "corn bread" is simply a representation of any amount.

**Practice Problem #2:**

**Rewrite $\frac{3}{8}$ as an equivalent common fraction whose denominator is 1,000.**

**Answer: $\frac{375}{1,000}$**

**Solution:**

**Any multiple of 8 must contain at least three factors of 2 (namely, in terms of prime factorization, $8 = 2 \times 2 \times 2$). 10 contains only one factor of 2 and 100 contains only two factors of 2. However 1,000 contains three factors of 2. In fact,**

$$1,000 = 8 \times 125.$$

**Hence we may multiply numerator and denominator of $\frac{3}{8}$ by 125 to obtain**

$$\frac{3}{8} = \frac{3 \times 125}{8 \times 125} = \frac{375}{1,000}.$$

**Notes:**

- **This problem is a special case of a general result that has rather far reaching implications. Namely, the only prime factors of powers of 10 are 2's and 5's. So as long as the denominator of a fraction contains only 2's and or 5's as prime factors we can always rewrite it as a fraction whose denominator is a power of 10. For example, in this problem we saw that $8 = 2 \times 2 \times 2$. Every time we multiply 2 by 5 we get 10. Hence to convert 8 into a power of 10, we must multiply it by 5 three times. That is:**

$$\begin{aligned}(2 \times 2 \times 2) \times (5 \times 5 \times 5) &= (2 \times 5) \times (2 \times 5) \times (2 \times 5) \\ &= 10 \times 10 \times 10 \\ &= 1,000\end{aligned}$$

**In order to be able to multiply the denominator of $\frac{3}{8}$ by 125 without changing the value of the fraction, we must also multiply the numerator by 125. In other words:**

$$\frac{3}{8} = \frac{3 \times 125}{8 \times 125} = \frac{375}{1,000}$$

**However a major problem occurs if the denominator of the fraction has a prime factor other than either 2 or 5. More specifically, such a fraction can never be equivalent to a common fraction whose denominator is a power of 10.**

**For example:**

**<u>Practice Problem #3</u>**

**Can $\frac{1}{3}$ ever be rewritten as an equivalent common fraction whose denominator is a power of 10? Explain your answer.**

**Answer: No**

**<u>Solution:</u>**

**Every power of 10 leaves a remainder of 1 when it is divided by 3. Hence there is never a whole number that we can multiply 3 by to obtain a power of 10. Thus there is no common fraction whose denominator is a power of 10 that is equivalent to $\frac{1}{3}$.[7]**

**<u>Note:</u>**

**To see why every power of 10 leaves a remainder of 1 when divided by 3, notice that the powers of 10 can be written in the form of a multiple of 9 with a remainder of 1; and 9 is a multiple of 3. More specifically:**

$$\begin{array}{rcl}
10 & = & 9+1 \rightarrow 10 \div 3 = 3\,\text{R1} \\
100 & = & 99+1 \rightarrow 100 \div 3 = 33\,\text{R1} \\
1{,}000 & = & 999+1 \rightarrow 1{,}000 \div 3 = 333\,\text{R1} \\
10{,}000 & = & 9{,}999+1 \rightarrow 10{,}000 \div 3 = 3{,}333\,\text{R1} \\
100{,}000 & = & 99.999+1 \rightarrow 100{,}000 \div 3 = 33{,}333\,\text{R1} \\
1{,}000{,}000 & = & 999{,}999+1 \rightarrow 1{,}000{,}000 \div 3 = 333.333\,\text{R1}, \text{etc}
\end{array}$$

[7]Recall that if we multiply the numerator and the denominator of a given fraction by the same non zero number we obtain a fraction that is equivalent to the given fraction. If two fractions are equivalent they are different names for the same rate.

## 3. A New "Fly In the Ointment"

**In place value notation, the digit furthest to the right names the denomination that is being named.; and up to now, it's understood that the digit furthest to the right names the ones place. Thus, for example, we may write 567 as an abbreviation for:**

| 100 | 10 | 1 |
|---|---|---|
| 5 | 6 | 7 |

**and if we wanted to abbreviate**

| 1,000 | 100 | 10 | 1 |
|---|---|---|---|
| 5 | 6 | 7 | |

**we'd use 0 as a place holder and write 5,670 to tell us that the 7 was holding the tens place rather than the ones place. In words, 5,670 is the "place value language" for saying "567 tens". To reemphasize a previous point, the important thing is that as long as we restrict our attention to whole numbers, the digit furthest to the right holds the ones place.**

**The problem occurs when we introduce the notion of decimal fractions[8]. For example, without using the word "tenths", how could we extend place value is that a person would know that we are talking about 567 tenths?**

**To see this in the form of a real life situation, we can think in terms of our own monetary system. In fact in addition to dimes and cents, let's also include an obsolete coin called a mill. The mill was used at a time when a cent had enough purchasing power that prices were thought of in terms of *mills*. [9] 10 mills were equal to 1 cent; and, therefore, 1,000 mills were worth a dollar. The word "mill" is derived from the Latin word "milla" which means a thousand.**

**So just as we nowadays write, for example, $27.49 to abbreviate**

| $10-bills | $1-bills | dimes | cents |
|---|---|---|---|
| 2 | 7 | 4 | 9 |

**in the "old" days, people would have written $27.493 as an abbreviation for**

| $10-bills | $1-bills | dimes | cents | mills |
|---|---|---|---|---|
| 2 | 7 | 4 | 9 | 3 |

[8]The term "decimal fraction" denotes those decimals that represent rational numbers. As we shall see later, decimals can represent numbers that are not rational..

[9]Even today, in states where the tax rate is based on per hundred rather than on per thousand dollars, to figure the taxes accurately, the tax rate is measured to the nearest mill rather than to the nearest cent.

**And if there had been a time when there was a denomination that was less than a mill and let's say we had called it a decimill (where 10 decimills = 1 mill), then we would have written $27.4931 as an abbreviation for**

| $10-bills | $1-bills | dimes | cents | mills | decimills |
|---|---|---|---|---|---|
| 2 | 7 | 4 | 9 | 3 | 1 |

**There is no need to limit our discussion to the development of a monetary system now; any more than it was necessary when we talked only about whole numbers. That is, we can generalize the above chart by writing**

| 10 | 1 | $\frac{1}{10}$ | $\frac{1}{100}$ | $\frac{1}{1,000}$ | $\frac{1}{10,000}$ |
|---|---|---|---|---|---|
| 2 | 7 | 4 | 9 | 3 | 1 |

**Notice that as long as the denominations are visible, there is no need to introduce a symbol to separate the whole number portion from the fractional portion of a number that's written in place value notation. As we shall see in the next section, the problem arises when we want to omit the names of the various denominations. for example**

**Practice Problem #4**

**Write 27 tenths as a common fraction whose denominator is 10**

**Answer:** $\frac{27}{10}$

**Solution:**

**27 tenths means 27 ÷ 10; which in the language of fractions is** $\frac{27}{10}$.[10]

**And to emphasize the adjective/noun theme we may write the answer in the form 27 tenths.**

**Note:**

**What is of interest to us now is how we can use place value notation to indicate that 27 is modifying tenths.**

[10]The mechanical way of obtaining the same result is to observe that since 27 is the adjective, it will be the numerator of our fraction and since “tenths” suggests 10, and 10 ends in one 0, our denominator will be a 1 followed by 1 zero. We will talk more about this soon.

**Practice Problem #5**

**Write 274 hundredths as a common fraction whose denominator is 1,000.**

**Answer: $\frac{2,740}{1,000}$**

**Solution:**

**We start just as we did in he previous problem.**

**Namely 274 hundredths means $274 \div 100$ and in the form of a common fraction this is $\frac{274}{100.}$ However, we want the fraction whose denominator is 1,000; and we obtain this by multiplying the numerator and denominator of $\frac{274}{100.}$ by 10 to obtain $\frac{2,740}{1,000}$.**

**Notes:**

- **The results we obtained in the above problem suggests a pattern, at least with respect to common fractions. Namely, observe that our adjective in this case is 274 and the denomination (noun) is hundredths. Since hundredth suggests 100 and 100 ends in 2 zeroes, the denominator is a 1 followed by 2 zeroes.**
- **This also happened in the previous illustration. In that case 27 was modifying tenths and 19 ends in 1 zero. Hence the denominator is a 1 followed by 1 zero, or $\frac{27}{10}$.**
- **This pattern continues in general. For example, to write 27 millionths as a common fraction, we observe that millionths suggests millions and in place value notation a million ends in 6 zeroes. Hence the denominator is a 1 followed by 6 zeroes. That is:**

$$\text{27 millionths} = \frac{27}{1,000,000}.$$

- **This discussion is continued in the next section.**

## 4. Another Place Holder: Introducing the Decimal Point

**Look at each line in the following chart:**

| | 10 | 1 | $\frac{1}{10}$ | $\frac{1}{100}$ | $\frac{1}{1,000}$ | $\frac{1}{10,000}$ | $\frac{1}{10,000}$ | $\frac{1}{1,000,000}$ |
|---|---|---|---|---|---|---|---|---|
| Line 1 | 2 | 7 | | | | | | |
| Line 2 | | 2 | 7 | | | | | |
| Line 3 | | | 2 | 7 | | | | |
| Line 4 | | | | 2 | 7 | | | |
| Line 5 | | | | | 2 | 7 | | |
| Line 6 | | | | | | 2 | 7 | |
| Line 7 | | | | | | | 2 | 7 |

**In each of the seven lines above, the adjective is 27. However, the noun that 27 is modifying depends on the column in which the 7 is located. For example, in Line 2 the 7 modifies tenths while in line 7 it modifies millionths.**

**The point is that as long as the denominations (nouns) are visible, we have no difficulty in seeing which denominations are modified by the 2 and the 7 However, suppose instead of seeing the above chart we saw a similar chart but with the denominations omitted. For example:**

| | | | |
|---|---|---|---|
| 2 | 7 | | |
| | | 2 | 7 |

**When we deal with whole numbers and see the numeral 27 we know that 27 is modifying "ones" because the digit furthest to the right holds the ones place. However with the advent of denominations that are less then 1, it is not clear what 27 is modifying. Thus the "challenge" becomes to find a way to modify the chart so that we can tell by looking at each line what noun the adjective 345 is modifying even though the denominations are not visible.**

**In other words, just as 0 is used as a place holder when we were dealing with whole numbers, we now need a different place holder to tell us where the whole number ends and the fractional part begins. In the chart below the place holder is represented by the "thick" line.**

**In terms of a mixed number interpretation, the "thick line" separates the whole number part from the fractional part of the number. The number to the left of the "thick line" is the whole number and the number to the right of the "thick line" is the fractional part.**

**Thus we would still read Line 1 either as 27 but we would read Line 2 either as 27 tenths (or as "2 and 7 tenths")[11] and the number on Line 7 would be read as 27 millionths.**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Line 1** | 2 | 7 | | | | | | |
| **Line 2** | | 2 | 7 | | | | | |
| **Line 3** | | | 2 | 7 | | | | |
| **Line 4** | | | | 2 | 7 | | | |
| **Line 5** | | | | | 2 | 7 | | |
| **Line 6** | | | | | | 2 | 7 | |
| **Line 7** | | | | | | | 2 | 7 |

**An easy way to remember the denomination that is determined by the "thick line" is to look at the number of digits to the right of it. For example, the first digit to the right of it represents $\frac{1}{10}$, and 10 ends in 1 zero. The second digit represents $\frac{1}{100}$ and 100 ends in 2 zeroes. In short with each place we go to the right of the "thick line", we annex one more 0 in the denominator.[12]**

**Notes:**

**• You are probably not used to seeing the "thick line" notation. And if this is the case it's because the decimal point, rather than the "thick line", was used to separate the whole number part of the number from the fractional part.**

**In other words, rather than write 3|45 we wrote instead of 3.45.**

**• Be careful with respect to symmetry. For example, tens and tenths are symmetric with respect to 1; not with respect to the decimal point. That is, the second digit to the left of 1 represents hundreds and the second digit to the right of 1 represents hundredths. However, the second digit to the left of the decimal point represents tens.[13]**

[11] It is common for people to look at, for example, 3.7 and read it as "3 point 7" and 3.45 as "3 point 45". "3 point 45" sounds like it should be greater than "3 point 7" because 45 is greater than 7. Remember that the decimal point determines the denomination that is being modified. "3.7" means "3 and 7 tenths" and "3.45" means "3 and 45 hundredths"; and 7 tenths is greater than 45 hundredths (in terms of money 7 dimes is worth more than 45 cents).

[12] This is analogous to what we do with the whole number denominations, that is, starting with 1, every time we move one more place to the left we annex one more 0. Thus the denomination that is four digits to the left of the ones place is 10,000 and the denomination that is four places to the right of the decimal point is $\frac{1}{10,000}$.

[13] A teacher once told me that when she taught decimals she told her students to say "one" when discussing whole numbers but to say "oneths" when discussing fractional parts.

**So let's revisit the decimal fraction 3.45 in terms of common fractions.**

- **If we pretend that the decimal point is not there, we see the whole number 345. This is the adjective; which in terms of common fractions means that the numerator is 345.**
- **Now count the number of digits that are to the right of the decimal point. In this case there are two (4 and 5). Hence the denomination is hundredths; which in terms of common fractions means that the denominator is 100.**
- **Therefore, written as a common fraction 3.45 is $\frac{345}{100}$ and is read as 345 *hundredths*.**

***<u>Notes</u>*:**

**-- 345 hundredths does not look like 3 and 45 hundredths but the two numerals are equivalent. More specifically, $\frac{345}{100}$ means $345 \div 100$ and as a mixed number this is $3\frac{45}{100}$, which is read as "3 and 45 hundredths".**

**-- An easy way to see the equivalence of 3.45 and $\frac{345}{100}$ is to think in terms of money. For example, we read $3.45 as "3 dollars and 45 cents". Since each dollar is worth 100 cents, we could give a person $3.45 by giving him 345 pennies; and since each penny is $\frac{1}{100}$ of a dollar, we could say that we gave the person $\frac{345}{100}$ dollars. We also sometimes write $3.45 as $\$3\frac{45}{100}$.**

**<u>Practice Problem #6</u>**

**Fill in the blank:**

**3.45 = ____ thousandths**

**Answer : 3,450**

**<u>Solution</u>*:***

**If we omit the decimal point in 3.45 we see that our adjective is 345. The fact that there are 2 digits to the right of the decimal point tells us that the noun 345 modifies is hundredths. In other words:**

$$3.45 = 345\text{ hundredths} = 345 \div 100 = \frac{345}{100}.$$

**However, the problem wants us to use thousandths as the noun; and in the language of common fractions, this means that our denominator has to be 1,000. To convert $\frac{345}{100}$ into an equivalent common fractions whose denominator is 1,000 we have to multiply both numerator and denominator by 10. That is:**

$$\frac{345}{100} = \frac{345 \times 10}{100 \times 10} = \frac{3{,}450}{1{,}000} = \text{3,450 thousandths}$$

***Aside*:**

**An interesting way to demonstrate the "adjective/noun" theme is to think in terms of probability. For example, to see the difference between, say 0.9 and 0.0009 [14] notice that in both 0.9 and 0.0009 the adjective is 9. However, since there is only one digit to the right of the decimal point in 0.9, the 9 modifies tenths. That is, $0.9 = \frac{9}{10}$. On the other hand, since there are four digits to the right of the decimal point in 0.0009; and since a 1 followed by four 0's is 10,000, the 9 modifies 10,000. That is, $0.0009 = \frac{9}{10{,}000}$.**

**-- So suppose there are 10 ping pong balls in a bag and 9 of them are colored red and the other one black. You make a bet that you will, without looking, randomly select one of the ping pong balls and that the ball will be red. Clearly you have 9 chances of picking a winner and since there are a total of 10 balls in the bag, we say that your chance of winning is 9 tenths and we write the probability as 0.9 or $\frac{9}{10}$.**

**-- Now suppose that there are 10,000 ping pong balls in the bag (it's a big bag!) and that 9 of them are still colored red but the rest are colored black. You again bet that you will, without looking, reach into the bag randomly and pick out a red ball. You still have the same 9 chances of winning the bet; but now, since there are 10,000 balls in the bag, you have 9,991 chances of losing. That is, your probability of winning is now 9 ten thousands which we write as 0.0009 or $\frac{9}{10{,}000}$.[15]**

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[14] In the decimal .9 it is easy to overlook the decimal point. It is rather small in appearance and it can easily be mistaken for a random blemish on the paper. So to help make sure that we notice that a decimal point is there, we place a 0 to the left of the decimal point. That is, rather than write .9 we often choose to write 0.9. There are several other conventions we follow in order to avoid confusion. For example, we often write 0 as Ø in order not to confuse the digit 0 with the letter O. Mathematicians often put a "bar" through the letter z in order not to confuse it with the numeral 2. That is they might write 3ƶ in order not to confuse it with the numeral 32. In many countries the numeral 1 is written more like the way in which we write the numeral 7. Therefore, in those countries they might write 7̶ to rather than 7 to denote the number 7 in a way that wouldn't make it easy to confuse it with the numeral they use to denote 1

[15] Closely connected with probability is the term "odds". If you have 9 chances of winning and 1 of losing (with all the outcomes being equally likely) we say that the odds in favor of your winning are 9 to 1. If instead you have 9,991 chances of losing we say that the odds against you are 9 to 9,991.

**Key Point:**

**The adjective part of a decimal fraction is the number we see if we omit the decimal point. On the other hand, the noun the decimal modifies is determined solely by the position of the decimal point.**

## 5. The Concept of Relative Size

**There are many forms for representing the same decimal fraction. For example, notice that when we talk about, say, ten dollars, it makes no difference whether we write $10 or $10.00. That is, in either case the 1 modifies the number of tens.[16] A good way to remember whether or not we need a zero is to notice whether inserting a zero changes the noun modified by the other digits. For example, 21.0000 means the same thing as 21 because in either representation the 2 and 1 modify ten and one respectively[17]. However, 0.021 and 0.21 represent different numbers because in the first case 2 is modifying hundredths while in the second case it's modifying tenths**

***Aside*:**

**Adding zeroes (or, equivalently, changing the noun the decimal fraction modifies) simply for effect can make an interesting psychological impact. For example, in the metric system a gram is the basic unit of weight. In this context, the weights 0.1 grams[18], 0.01 grams and 0.001 grams are given the names, 1 decigram, 1 centigram and 1 milligram respectively. A gram is a very small amount of weight. In fact it takes 454 grams to equal 1 pound; and since a milligram is 0.001 grams, it requires 454,000 milligrams to equal 1 pound.**

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[16]We could also have represented ten dollars as $10.0 or $10.000 etc. However, when it comes to writing dollar amounts in decimal form it is conventional to write to the nearest cent. that is, we usually write $3.70 when it would have been equally correct to have written $3.7.

[17]Scientists and engineers talk about *significant figures*. Namely in mathematics we can say that a piece of string is exactly 6 inches long but in the "real world" measurements are not exact. Thus the scientist will write 6.0 inches to let us know that the measurement is guaranteed to be accurate to the nearest tenth of an inch; but he'll write 6.00 inches if he wants to tell us that the measurement is guaranteed to be accurate to the nearest hundredth of an inch.

[18]0.1 and .1 mean the same thing. However by writing the 0 to the left of the 1, it emphasizes the existence of the decimal point. In other words, when one sees .6 one can wonder whether the "dot" is just a random mark on the paper or a decimal point. However when we write 0.6, it's clear that the "dot" is a decimal point.

**Because many drugs are very toxic, small amounts can be very dangerous. For that reason, we measure drugs in milligrams (mgs). Thus, rather than talk about 0.1 grams of a particular drug, we would refer to it as 100 mgs.[19] Similarly, we would refer to a dosage of 0.2 grams as 200 mgs. While the difference between 0.1 and 0.2 grams seems negligible, the difference between 100 mgs and 200 mgs seems non-negligible. In this context the casual user might think that there is little difference between taking 0.1 grams or 0.2 grams but they might think twice about taking 200 mgs rather than 100 mgs.[20]**

**<u>Practice Problem #7</u>**

**Which of the following two decimal fractions names the greater rational number, 0.120999 or 0.121100?**

***Answer:*** 0.012110

**<u>Solution</u>*:***

**The traditional way is to look at the two decimal fractions denomination by denomination until we come to the first place where the digits are different; in which case the fraction with the greater digit is the greater. Thus in the present example**

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | ↓ | | | |
| 0. | 1 | 2 | 0 | 9 | 9 | 9 |
| 0. | 1 | 2 | 1 | 1 | 0 | 0 |
| | | | ↑ | | | |

**Since 0 < 1; 0.120999 is less than 0.121100.**

[19]That is, 0.1 grams is the same as 0.100 grams and since the 0 furthest to the left is in the thousandths place, 0.100 is the same as 100 mgs.

[20]The same logic can be applied in reverse. If, for example, the government talks about a deficit of $1,200,000,000,000, the average "concerned citizen" might be quite alarmed. However, if the government decides to write the deficit in the form $1.2 trillion, the adjective 1.2 doesn't seem to be very large and the word "trillion" masks the denomination quite successfully. That is, to most people "trillion" and "billion are "just words"!

**<u>Notes</u>:**

**• The adjective/noun theme works nicely here and eliminates our having to think in terms of fractions. Namely, without the decimal points the numbers we would see are 120,999 and 121,100 and since there are six digits to the right of the decimal point in each number, the two decimal fractions modify "millionths". In other words:**

**0.120999 $=$ 120,999 millionths**

**0.121100 $=$ 121,100 millionths**

**And whenever 120,999 and 121,100 modify the same noun, 121,100 $>$ $>$ 120,999**

**• More generally comparing the size of two decimal fractions can always be replaced by comparing an equivalent pair of whole numbers. In particular, if both decimal fractions have the same number of digits to the right of the decimal point they modify the same noun. So, for example, to compare, say 0.01 with 0.000998, notice that 0.01 has only 2 digits to the right of the decimal point while 0.000998 has 6 digits to the right of the decimal point. Therefore we rewrite 0,01 as 0.010000. In that case, 0.01 represents 10,000 millionths and 0.00998 represents 998 millionths. Since 10,000 is greater than 998, 10,000 millionths is greater than 998 millionths. In other words 0.01 is greater than 0.000998.**

**Practice Problem #8**

**Which of the following two decimal fractions names the greater rational number, 0.53 or 0.491?**

**Answer: 0.53**

**Solution*:***

**0.53 means 53 hundredths and 0.491 means 491 thousandths. We cannot compare the adjectives unless they are modifying the same noun.**

**To this end we may rewrite 0.53 as 0.530. Then:**

**0.53 = 530 thousandths**

**And 530 thousandths is great than 491 thousandths**

**0.491 = 491 thousandths**

***Note*:**

**As we mentioned in a previous note, there is a tendency on the part of many people to read 0.53 as "point 53" and 0.491 as "point 491". In this context, 491 seems greater than 53. We have to keep track of what 53 and 491 modify (in much the same way that 17 is greater than 11 but 17 pennies is less than 1 quarter).**

**This completes our introduction to decimals.**