Arithmetic Revisited

Lesson 4:

Part 5: Application to Percents

1. Introduction to Percents

The fact that people like to think in terms of powers of ten[1] makes the use of mixed numbers very important. Suppose, for example, a teacher gives a test with 5 equally weighted problems and a student gets 4 answers correct. The teacher writes 4/5 to indicate that the student got 4 right out of a possible 5. In other words, the grade says "On a scale of 0 to 5, you got 4". In terms of common fractions, what 4/5 means is that whatever number denotes a perfect score, your grade is $\frac{4}{5}$ of it. The usual "perfect score" is 100 and $\frac{4}{5}$ of 100 is 80. In this case we write the score as 80%, which is read as 80 *percent*,. which means that on a scale of 100, the score is 80.

***Historical Aside*:**

The word "percent" means "per hundred"; and in terms of arithmetic this means "$\div$100". In fraction notation, the 100 would be the denominator. Prior to type setting it was common to use a slash mark (/) rather than the horizontal fraction bar[2]. Thus 80% would be written as 80/100. As an abbreviation, the slash mark was placed between the two 0's and the 1 was omitted. Thus, 80/100 became 80 0/0, which in turn became 80 %.

No matter what a perfect score is, to find $\frac{4}{5}$ of it we would divide the perfect score by 5 and then multiply the answer by 4. In this context, as long as the perfect score is a multiple of 5, there is no need to "invent" mixed numbers. With this in mind, let's suppose now that there were 7 equally weighted problems on the test and that the student solved 6 of these problems correctly. The teacher would now write 6/7 on the student's paper. This would mean that on a scale of 0 to 7, the student earned a grade of 6. However, the grade is usually recorded on a scale of 0 to 100.

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[1]Notice how often we talk about "on a scale of 1 to 10" or when we want to talk about something that is as likely to fail as to succeed we talk about a "50 - 50" proposition. When a team wins half of its games we refer to it as "500 team" because 500 is half of 1,000.

[2]In other words, rather than type or write $\frac{4}{5}$, people would write 4/5.

The problem is that 100 is not a multiple of 7. In fact the two multiples of 7 that are closest to 100 are 98 and 105. More specifically, $14 \times 7 = 98$ and $15 \times 7 = 105$. Hence, the fact that

$$\frac{6}{7} = \frac{6 \times 14}{7 \times 14} = \frac{84}{98}$$

means that on a scale from 0 to 98, 6/7 is equal to 84

and the fact that

$$\frac{6}{7} = \frac{6 \times 15}{7 \times 15} = \frac{90}{105}$$

means that on a scale from 0 to 105, 6/7 is equal to 90. In this case we do not say either 84% or 90%. The percent sign is reserved for use only when the perfect score is based on a scale from 0 to 100. Since 100 is greater than 98 but less than 105, we can conclude that the exact score is greater than 84% but less than 90%.

- In many cases, knowing that your grade is more than 84% but less than 90% is sufficient information to conclude that your grade is a B. On the other hand, there are times when a difference of 1 point (for example in competing for an award) can be quite significant.

- In this context, there are times when we are willing to settle for knowing that our grade is between 84 and 90 and there are other times when we want a more precise answer.

- The above observation applies to situations other than grades. Remember that fractions are adjectives. For example, 1% represents a rate of 1 per hundred whether we take 1% of \$100 or 1% of \$100,000,000, However in the case of \$100, 1% represents only \$1 and we might feel that this is an insignificant amount. On the other hand, 1% of \$100,000,000 represents \$1,000,000, which is an amount we might not be quite as willing to call "insignificant".

To find the exact answer we take advantage of the fact that $\frac{6}{7}$ of 100 means $\frac{6}{7} \times 100$. We then see that:

$$\frac{6}{7} \times \frac{100}{1}\% =$$

$$\frac{600\%}{7} =$$

$$600\% \div 7 =$$

$$85\,\text{R}5 = 85\tfrac{5}{7}\%.$$

In other words 6 on a scale of 0 to 7 is the same rate as $85\frac{5}{7}$ on a scale of 0 to 100. If the teacher rounds off you grade to 86 you are receiving $\frac{2}{7}$ of 1% (that is, $\frac{2}{7}$ of a point) more than you deserve but if (s)he rounds it down to 85 you are being "cheated" out of $\frac{5}{7}$ of a point.

- Of course, there's no need to restrict the meaning of $\frac{6}{7}$ of 100% to a test score. For example, if you own $\frac{6}{7}$ of a business; in the language of percents, you own exactly $85\frac{5}{7}$% of the business. In this situation, if you settled for owning 85% of the business you'd be losing $\frac{5}{7}$ of 1% of the profit. This is not a large percent but if the business made a profit of $700 million, 1% of it is 7 million and therefore $\frac{5}{7}$ of 1% represents $5 million.

2. The Corn Bread Model:

It is traditional to have students memorize some conversions between fractions (including mixed numbers) and percents. For example, $\frac{1}{2} = 50\%$; $\frac{1}{4} = 25\%$; $\frac{1}{3} = 33\frac{1}{3}\%$, etc. However our above method for converting $\frac{6}{7}$ into an equivalent percent, works for any common fraction.

Rather than use an abstract algorithm, let's return to our "corn bread" model. However in terms of this model, the study of percents involves corn breads that come pre-sliced into 100 equally sized pieces. In other words, in this model 1% means the same thing as 1 piece of the corn bread.

-- This presents a problem in the sense that the only divisors of 100, are 1, 2, 4, 5, 10, 20, 25, 40, 50 and 100. So suppose, for example, that we wanted to share 6 corn breads equally among 7 people, Ordinarily if there were 6 corn breads and 7 people we would cut each corn bread into 7 equally-sized pieces and give each of the 7 people 1 piece form each of the 6 corn breads. .

-- However we can't do that because the corn bread has already been sliced into 100 equally sized pieces. If we decide to think in terms of "pieces" rather than in terms of whole corn breads, we see that we may think of the 6 corn breads as being 600 pieces. $600 \div 7 = 85$ R5. Thus we can give each person 85 pieces, after which we have 5 pieces left .

-- We can then divide each of these 5 remaining pieces into 7 pieces of equal size so that each person gets an additional 5 pieces of a corn bread. Thus each person $85\frac{5}{7}$ pieces.

-- Since each piece is 1% of the corn bread, each person gets $85\frac{5}{7}\%$ of a corn bread.

.-- Translating what we did with the corn breads into more formal language, what we did was to view the problem as follows:

$$\begin{aligned}\tfrac{6}{7}\text{ of “the whole”} &= \tfrac{6}{7}\text{ of }100\% \\ &= \tfrac{6}{7}\times 100\% \\ &= \tfrac{600}{7}\% \\ &= 85\tfrac{5}{7}\%\end{aligned}$$

Practice Problem #1

What percent of a business do you own if you own $\frac{6}{13}$ of it?

Answer: **$46\frac{2}{13}$**

Solution

$$\begin{aligned}\tfrac{6}{13}\text{ of “the whole”} &= \tfrac{6}{13}\text{ of }100\% \\ &= \tfrac{6}{13}\times 100\% \\ &= \tfrac{600}{13}\% \\ &= 46\tfrac{2}{13}\%\end{aligned}$$

Notes:

- **Since $\frac{6}{12}$ is half of the whole, or 50%, and since $\frac{6}{13}$ is a “little less” than $\frac{6}{12}$, we should expect an answer that is a “little less” than 50%. Hence $46\frac{2}{13}\%$ is a plausible answer.**

- **In terms of our corn bread model, if the corn bread is pre-sliced into 100 equally sized pieces, $46\frac{2}{13}\%$ of the corn bread is $\frac{6}{13}$ of 100 pieces.**

- **From a slightly different perspective,**

$$\tfrac{1}{13}\text{ of }100 = 100 \div 13 = 7\tfrac{9}{13}$$

In other words, the corn bread may now be viewed as being divided into 13 parts of equal size; and each part is $7\frac{9}{13}$ of the 100 pieces. Thus:

$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$	$7\frac{9}{13}$

The shaded region represents $6\times(7+\frac{9}{13})$ or $42\frac{54}{13}$ pieces; and, in turn this is equal to $42+(4+\frac{2}{13}) = 46\frac{2}{13}$

- **The fact that a very small error in computing a percentage[3] exactly can lead to a huge error "down the road" makes it important that we know how to compute the exact value of such fractions as $85\frac{5}{7}$%. Moreover, most people are more comfortable with such statements (ratios) as "6 out of every 7 people......" rather than with such statements as "$85\frac{5}{7}$% of the population....."**

3. Converting Percents to Common Fractions

In the previous section we started with $\frac{6}{7}$ and showed how to convert it to $85\frac{5}{7}$%. In this section we want to start with $85\frac{5}{7}$% and show how to convert it to an equivalent common fraction. There are several different ways to do. For example:

- **Recall that $85\frac{5}{7}$% means $85\frac{5}{7}$ per 100. This in turn means $85\frac{5}{7} \div 100$**

$$
\begin{aligned}
85\tfrac{5}{7}\% &= 85\tfrac{5}{7} \text{ per } 100 \\
&= 85\tfrac{5}{7} \div 100 \\
&= (85\tfrac{5}{7} \times 7) \div (100 \times 7) \\
&= [(85 \times 7 + (\tfrac{5}{7} \times 7)] \div 700 \\
&= (595 + 5) \div 700 \\
&= 600 \div 700 \\
&= \tfrac{6}{7}
\end{aligned}
$$

[3]There is a tendency to use "percent" and "percentage" interchangeably even though, rigorously speaking they are different. In terms of our overall theme, percent denotes what fractional part we're taking while percentage is the numerical value of the fractional part. For example, in the statement 50% of 150 $= 75$, the percent is 50 and the percentage is 75.

- **The above chart can be translated into fractional notation. That is:**

$$
\begin{aligned}
85\tfrac{5}{7}\% &= \\
85\tfrac{5}{7} \text{ per } 100 &= \\
85\tfrac{5}{7} \div 100 &= \\
\frac{85\frac{5}{7}}{100} &= \\
\frac{85\frac{5}{7} \times 7}{100 \times 7} &= \\
\frac{595 + 5}{700} &= \\
\frac{600}{700} &= \\
\frac{6}{7} &
\end{aligned}
$$

In more "telescopic" terms, a shorter version would have been to rewrite $85\frac{5}{7}\%$ as an equivalent fraction ($\frac{600}{7}$); then "annex" two 0's to the denominator ($\frac{600}{700}$); and finally, if desired, reduce the fraction to lowest terms ($\frac{6}{7}$).

***<u>A Special Note</u>*:**

Students often confuse "$\frac{6}{7}$ of the whole" with "$\frac{6}{7}\%$ of the whole". $\frac{6}{7}\%$ means $\frac{6}{7}$ of 1% and this is clearly less than 1%. To convert a fractional part of 1% to a fraction, we simply remove the percent sign and annex two 0's to the denominator. Thus "$\frac{6}{7}\%$ of the whole" means the same thing as "$\frac{6}{700}$ of the whole" (*<u>not</u>* $\frac{6}{7}$ of the whole). One way to illustrate this is the same way as we did with converting $85\frac{5}{7}\%$ to a fraction. Namely:

$$
\begin{aligned}
\tfrac{6}{7}\% &= \tfrac{6}{7} \text{ per } 100 \\
&= \tfrac{6}{7} \div 100 \\
&= (\tfrac{6}{7} \times 7) \div (100 \times 7) \\
&= 6 \div 700 \\
&= \tfrac{6}{700}
\end{aligned}
$$

Practice Problem #2

Write $7\frac{2}{3}\%$ as a common fraction in lowest terms.

Answer: $\frac{23}{300}$

Solution:

$$
\begin{aligned}
7\tfrac{2}{3}\% &= \\
7\tfrac{2}{3} \text{ per } 100 &= \\
7\tfrac{2}{3} \div 100 &= \\
\frac{7\frac{2}{3}}{100} &= \\
\frac{7\frac{2}{3} \times 3}{100 \times 3} &= \\
\frac{21+2}{300} &= \\
\frac{23}{300}
\end{aligned}
$$

Notes:

- **We could first convert $7\frac{2}{3}\%$ to an improper fraction to obtain**

$$\begin{aligned}
7\tfrac{2}{3}\% &= \\
7\tfrac{2}{3} \text{ per } 100 &= \\
7\tfrac{2}{3} \div 100 &= \\
\tfrac{23}{3} \div 100 &= \\
\tfrac{23}{3} \times \tfrac{1}{100} &= \\
\tfrac{23\times 1}{3\times 100} &= \\
\tfrac{23}{300}
\end{aligned}$$

- **In doing the above computation, in effect we simply wrote the mixed number as an improper fraction and "annexed" two 0's to the denominator.**

- **In terms of a practical example, suppose your boss announces that he is giving you a $7\frac{2}{3}\%$ raise. Then expressed in terms of whole numbers, you will be getting an additional \$23 for every \$300 you were earning before the raise.**

This completes our section on mixed numbers and percents. Additional practice is supplied in the next Lesson.