

****Unedited Draft****

Arithmetic Revisited

Lesson 4:

Part 4: Dividing Mixed Numbers

1. Converting the Numbers Into Improper Fractions

A nice way to solve a new problem is to convert it into an equivalent problem that we already know how to solve. In this respect, since we already know how to divide fractions, it would seem logical to divide mixed numbers by first converting them into improper fractions. For example, suppose we want to divide $4\frac{1}{2}$ by $2\frac{1}{2}$. We could begin by rewriting $4\frac{1}{2} \div 2\frac{1}{2}$ as $\frac{9}{2} \div \frac{5}{2}$. We would then use the “invert and multiply” rule to obtain $\frac{9}{2} \times \frac{2}{5}$, or $\frac{9}{5}$. Finally we divide 9 by 5 and obtain $1\frac{4}{5}$ as the answer.

Notes:

- The definition of division hasn't changed. $4\frac{1}{2} \div 2\frac{1}{2}$ means the number we must multiply by $2\frac{1}{2}$ to obtain $4\frac{1}{2}$ as the product. A quick check shows that:

$$2\frac{1}{2} \times 1\frac{4}{5} = \frac{5}{2} \times \frac{9}{5} = \frac{9}{2} = 4\frac{1}{2}.$$

- It is often helpful to estimate an answer before proceeding with the actual computation. Noticing that $4\frac{1}{2}$ is “around” 4 and $2\frac{1}{2}$ is “around” 2, we can estimate that the answer should be “around” $4 \div 2$ or 2. Actually since $5 \div 2\frac{1}{2}$ is exactly 2, $4\frac{1}{2} \div 2\frac{1}{2}$ must be a “little less” than 2. Thus, $1\frac{4}{5}$ is a plausible answer.
- What the quotient tells us is that not only is $4\frac{1}{2}$ almost twice as much as $2\frac{1}{2}$ but it's exactly $1\frac{4}{5}$ times as much.
- As a practical application, suppose we can buy $2\frac{1}{2}$ pounds of cheese for $\$4\frac{1}{2}$. Then the price per pound (that is, “dollars per pound”) is obtained by dividing $\$4\frac{1}{2}$ by $2\frac{1}{2}$ pounds. The quotient tells us that the cost of the cheese is $\$1\frac{4}{5}$ or \$1.80 per pound.

- As a check notice that \$1.80 per pound, 2 pounds would cost \$3.60 and $\frac{1}{2}$ pound would cost \$0.90; and we then see that the total cost is $\$3.60 + \0.90 or \$4.50

Practice Problem #1

Use the above method to express $6\frac{1}{3} \div 1\frac{3}{4}$ as a mixed number.

Answer: $3\frac{13}{21}$

Solution:

$$\begin{aligned}
 6\frac{1}{3} \div 1\frac{3}{4} &= \frac{19}{3} \div \frac{7}{4} \\
 &= \frac{19}{3} \times \frac{4}{7} \\
 &= \frac{76}{21} \\
 &= 76 \div 21 \\
 &= 3\frac{13}{21}
 \end{aligned}$$

Notes:

- To check that our answer is at least reasonable, we observe that rounded off to the nearest whole number $6\frac{1}{3} = 6$ and $1\frac{3}{4} = 2$. Hence our answer should be “reasonably close to” $6 \div 2$ or 3.
- Once we obtain our answer, we can check to see if it's correct by remembering that $6\frac{1}{3} \div 1\frac{3}{4} = 31\frac{13}{21}$ means that $31\frac{13}{21} \times 1\frac{3}{4} = 6\frac{1}{3}$. Thus to check our answer we compute the value of $31\frac{13}{21} \times 1\frac{3}{4}$ and verify that it is equal to $6\frac{1}{3}$

$$\begin{aligned}
 31\frac{13}{21} \times 1\frac{3}{4} &= \frac{76}{21} \times \frac{7}{4} \\
 &= \frac{76 \times 7}{21 \times 4} \\
 &= \frac{2 \times 2 \times 19 \times 7}{3 \times 7 \times 2 \times 2} \\
 &= \frac{2 \times 2 \times 19 \times 7}{3 \times 7 \times 2 \times 2} \\
 &= \frac{19}{3} \\
 &= 6\frac{1}{3}
 \end{aligned}$$

2. Using the Mixed Numbers as Adjectives Modifying the Same Noun.

To divide $4\frac{1}{2}$ by $2\frac{1}{2}$, we may think of both mixed numbers as modifying, say, “a carton of books” where each carton contains 2 books¹. In this case, $4\frac{1}{2}$ cartons is another name for 9 books and $2\frac{1}{2}$ cartons is another name for 5 books. Hence we may visualize the problem in terms of the following steps:

$$\begin{aligned} 4\frac{1}{2} \div 2\frac{1}{2} &= 4\frac{1}{2} \text{ cartons} \div 2\frac{1}{2} \text{ cartons} \\ &= 9 \text{ books} \div 5 \text{ books} \\ &= 9 \div 5 \\ &= 1\frac{4}{5} \end{aligned}$$

While converting mixed numbers to improper fractions gives us the desired result when we divide two mixed numbers, from a mathematical perspective, it would be nice to know that the arithmetic of mixed numbers is self-contained (at least in the sense that we aren't forced to rewrite mixed numbers as improper fraction in order to do arithmetic). From a teaching point of view, we might want to illustrate how we can divide mixed numbers in ways that may be more intuitive to students. In this context:

Multiplying Both Numbers in a Ratio by the Same (Non Zero) Number:

Students might find it interesting to see that we can divide mixed numbers by “translating” the problem into an equivalent whole number problem without having to refer to such things as books and cartons. The key is that it is still a fact that we do not change a quotient when we multiply both the dividend and the divisor by the same (non zero) whole number. To this end, if we are given the division problem $4\frac{1}{2} \div 2\frac{1}{2}$ we simply multiply both numbers by 2 to obtain:

$$\begin{aligned} 4\frac{1}{2} \div 2\frac{1}{2} &= (4\frac{1}{2} \times 2) \div (2\frac{1}{2} \times 2) \\ &= 9 \div 5 \\ &= 1\frac{4}{5} \end{aligned}$$

¹More generally we would use the (least) common multiple of both denominators. For example, had the problem been $4\frac{1}{3} \div 2\frac{5}{7}$, we would have assumed that each carton contained 21 books and then multiplied both the dividend and the divisor by 21.

Practice Problem #2

Use the above method to write $6\frac{1}{3} \div 2\frac{3}{5}$ as a mixed number.

Answer: $2\frac{17}{39}$

Solution:

Since the denominators are 3 and 5, we can eliminate them by multiplying both numbers by 15.

$$\begin{aligned} 15 \times 6\frac{1}{3} &= 15 \times (6 + \frac{1}{3}) \\ &= (15 \times 6) + (15 \times \frac{1}{3}) \\ &= 90 + 5 \\ &= 95 \end{aligned}$$

$$\begin{aligned} 15 \times 2\frac{3}{5} &= 15 \times (2 + \frac{3}{5}) \\ &= (15 \times 2) + (15 \times \frac{3}{5}) \\ &= 30 + 9 \\ &= 39 \end{aligned}$$

$$\text{Hence } 6\frac{1}{3} \div 2\frac{3}{5} = 95 \div 39 = 2\frac{17}{39}$$

As a check we see that

$$\begin{aligned} 2\frac{17}{39} \times 2\frac{3}{5} &= \frac{95}{39} \times \frac{13}{5} \\ &= \frac{95 \times 13}{39 \times 5} \\ &= \frac{19 \times 5 \times 13}{3 \times 13 \times 5} \\ &= \frac{19 \times 5 \times 13}{3 \times 13 \times 5} \\ &= \frac{19}{3} \\ &= 6\frac{1}{3} \end{aligned}$$

²In terms of relative size, the above result tells us that $6\frac{1}{3}$ is approximately $2\frac{1}{2}$ times as great as $2\frac{3}{5}$ (It is actually $2\frac{17}{39}$ times as great)

An Interesting Aside:

We don't usually think about using common denominators when we want to divide two fractions but we can. That is, the denominator of a fraction is the noun and when we divide two numbers that modify the same noun, the nouns "cancel". Thus if we decide to rewrite the mixed numbers as improper fractions we obtain:

$$\begin{aligned}
 6\frac{1}{3} \div 2\frac{3}{5} &= \frac{19}{3} \div \frac{13}{5} \\
 &= \frac{19 \times 5}{3 \times 5} \div \frac{13 \times 3}{5 \times 3} \\
 &= \frac{95}{15} \div \frac{39}{15} \\
 &= 95 \text{ fifteenths} \div 39 \text{ fifteenths} \\
 &= 95 \div 39 \\
 &= 2\frac{17}{39}
 \end{aligned}$$

Enrichment Note

While the topic might be beyond the scope of the elementary school student, it is interesting to note that the long division algorithm for whole numbers, as a form of repeated subtraction, also applies to the division of mixed number. For example,

$$6\frac{1}{3} \div 2\frac{3}{5} = (\quad)$$

means the same thing as

$$2\frac{3}{5} \times (\quad) = 6\frac{1}{3}$$

Using trial-and error to "fill in the blank" we see that

$$2\frac{3}{5} \times 1 = 2\frac{3}{5}$$

$$2\frac{3}{5} \times 2 = 4\frac{6}{5} = 5\frac{1}{5} \quad \leftarrow \quad \text{less than } 6\frac{1}{3}$$

$$2\frac{3}{5} \times 3 = 6\frac{9}{5} = 7\frac{43}{5} \quad \leftarrow \quad \text{greater than } 6\frac{1}{3}$$

← { In other words, $6\frac{1}{3}$ is more than twice as big as $2\frac{3}{5}$ but less than three times as big.

The difference between $6\frac{1}{3}$ and $5\frac{1}{5}$ is $6\frac{5}{15} - 5\frac{3}{15} = 1\frac{2}{15}$. Hence if we use the long division algorithm we see that:

$$\begin{array}{r} 2 \quad \text{R} = 1\frac{2}{15} \\ 2\frac{3}{5} \overline{) 6\frac{1}{3}} \\ - \quad 5\frac{1}{5} \\ \hline 1\frac{2}{15} \end{array}$$

and if we now write the remainder over the divisor we see that

$$6\frac{1}{3} \div 2\frac{3}{5} = 2 + \frac{1\frac{2}{15}}{2\frac{3}{5}}$$

We can simplify the complex fraction $\frac{1\frac{2}{15}}{2\frac{3}{5}}$ by multiplying numerator and denominator by 15. That is

$$15 \times (1\frac{2}{15}) = 15 \times (1 + \frac{2}{15}) = 15 + 2 = 17$$

and

$$15 \times (2\frac{3}{5}) = 15 \times (2 + \frac{3}{5}) = 30 + 9 = 39$$

Hence

$$6\frac{1}{3} \div 2\frac{3}{5} = 2 + \frac{1\frac{2}{15}}{2\frac{3}{5}} = 2 + \frac{17}{39} = 2\frac{17}{39}$$