Unedited Draft

Arithmetic Revisited

Lesson 4:

Part 1: Introduction to Mixed Numbers

1. Prelude to Mixed Numbers

The best way to comprehend a new topic is by discovering that it is simply a different form of what you already know. This idea extends to how we can introduce mixed numbers to our students in a natural (and relevant) way in terms of our "adjective/noun" theme. For example, long before students become aware of such numerals as $5\frac{7}{12}$ they have probably come into contact with such expressions as:

- **-- a person being 5 feet 7 inches tall.**
- **-- or a trip that took 3 hours and 17 minutes.**
- **-- or a cut of meat that weighed 4 pounds and 3 ounces.**

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Notes:

• In this context, we have "mixed together" two nouns (feet and inches, hours and minute, pounds and ounces, etc.)) and modified each with a whole number adjective.

• if we wish to express 5 feet 7 inches using only one noun, we can either express it in inches or in feet. If we elect to use inches, we use the fact that since there are 12 inches in a foot, there are 60 inches in 5 feet. Hence in 5 feet 7 inches there are 67 inches.

• if we prefer to use feet as the noun, we have 5 "whole" feet plus 7 of the 12 inches that are needed for an additional foot. Expressed as a fraction "7 of what it takes 12 of to make 1 foot" is $\frac{7}{12}$ feet. Hence we may express 5 feet 7 inches in the form $(5+\frac{7}{12})$ feet.

• We agree to omit the plus sign and abbreviate $5+\frac{7}{12}$ by writing $5\frac{7}{12}$.

• That is, while $5\frac{7}{12} \neq 67$, $5\frac{7}{12}$ *feet* $= 67$ *inches*.

• Although it is mathematically correct to rewrite 5 feet 7 inches as 4 feet 19 inches, and, therefore, to write $5\frac{7}{12}$ feet as $4\frac{19}{12}$ feet; the

convention is that we do not usually do that.[1] More specifically we define a mixed number as being the sum of a whole number plus a rational number that is less than 1.[2]

- **In a similar way if we wish to express 3 hours and 17 inches using only one noun, we can either express the time in hours or in minutes. If we elect to use minutes, we use the fact that since there are 60 minutes an hour, there are 180 minutes in 3 hours. Hence in 3 hours 17 minutes there are 197 inches.**
- **if we prefer to use hours as the noun, we have 3 "whole" hours plus 17 of the 60 minutes that are needed for an additional hour. Expressed as a fraction "17 of what it takes 60 of to make 1 hour" is $\frac{17}{60}$ feet. Hence we may express 3 feet 17 minutes in the form $(3+\frac{17}{60})$ hours.**
- **We agree to omit the plus sign and abbreviate $3+\frac{17}{60}$ by writing $3\frac{17}{60}$.**
- **That is, while $3\frac{17}{60} \neq 197$, $3\frac{17}{60}$ *hours* $=$ 197 *minutes*.**
- **Using similar reasoning, the fact that 16 ounces $=$ 1 pound means that 4 pounds 3 ounces is equal to either 67 ounces or $4\frac{3}{16}$ pounds.**

2. Relating Mixed Numbers and Improper Fractions .

As we saw in the previous lesson, the concept of a rational number was introduced almost simultaneously with our discussion about the type of answer we get when we divide one whole number by another. More specifically, we defined a rational number to be any number that can be expressed as the quotient of 2 whole numbers. To illustrate the existence of a rational number that was not a whole number, we talked about the quotient $2 \div 3$ or $\frac{2}{3}$; which we viewed as the answer to a question such as "How much corn bread does each, person get if 2 equally sized corn breads are shared equally by 3 people?"

[1]This corresponds with the idea of getting good estimates. For example we could say that the height of a person was 3 feet 37 inches. However, it gives us an easier to estimate if we say 6 feet 1 inch. In a similar way if we are packing books in cartons that hold 12 books each we don't usually say that we have 3 full cartons and 37 books left over. Namely we keep packing cartons until there aren't enough books left to fill another carton.

[2]When we use a common fraction to represent rational numbers, we call the common fraction a *proper* fraction if it represents a rational number that is less than 1. In this context, the mathematical definition of a mixed number is that it is the sum of a whole number and a proper fraction.

However if there were 200 cornbreads instead of 2 cornbreads, it would be very tedious to take the time to slice each of the 200 cornbreads into 3 equally sized pieces. In fact, even if the cornbreads had already been sliced this way, it would still be very tedious to "deal" the pieces of cornbread, one at a time, to the 3 people. Most likely,we would revert to our knowledge of whole number division and compute that $200 \div 3 = 66$ "with 2 left over".

Hence if we wanted to use the least amount of effort to distribute the cornbreads, we would give each person 66 cornbreads and we would then divide the remaining 2 cornbreads among the 3 people by giving each person $\frac{2}{3}$ of a cornbread.

Thus we could either:

(1) slice each of the 200 pieces of equal size and give each person 1 piece from each of the 200 cornbreads. In that way, each person would get 200 pieces or 200 "of what it takes 3 of to equal 1 corn bread". In other words each person would get 200 thirds of corn bread; or $\frac{200}{3}$ cornbreads

or

(2) give each of the persons 66 corn breads plus 1 piece from each of the 2 remaining corn breads. In this way each person receives $66\frac{2}{3}$ corn breads.

***Notes*:**

- **For most people, $66\frac{2}{3}$ corn breads tells us more explicitly how many cornbreads each person got. Writing $\frac{200}{3}$ cornbreads gives us the same information implicitly. That is, we have to do some computation to determine how many cornbreads is represented by "200 of what it takes 3 of to make a whole cornbread". However, while it might seem a bit awkward, there is nothing wrong with saying that each person got $\frac{200}{3}$ cornbreads.**

- **We often refer to $\frac{200}{3}$ as an *improper* fraction. More generally, the common fraction $\frac{n}{m}$ is called an improper fraction if $n \geq m$. The reason is that an improper fraction can always be expressed as the sum of a whole number and a proper fraction (i.e., a common fraction that is less than 1).[3]**

[3] It would be more polite to refer to an improper fraction as, perhaps, a "top heavy" fraction. - However there is nothing that is really "improper" about an improper fraction. For example if a recipe calls for $2\frac{1}{3}$ cups of flour and you can only find your "$\frac{1}{3}$ of a cup" measuring cup, all you have to remember is that

In the same way that we could begin the study of fractions almost simultaneously with the study of division, we could also introduce mixed numbers.[4] For example:

Practice Problem #1

How many cornbreads does each person get if 38 cornbreads are to be shared equally among 7 persons?

Answer: $7\frac{3}{7}$ (or $\frac{38}{7}$)

Solution*:*

If we want to solve this problem in the way we did in our lesson on common fractions, we could pre-slice each of the 38 cornbreads into 7 equally sized pieces and deal out 1 piece from each of the cornbreads to each of the 7 people. In this way, each person gets 38 pieces of cornbread; that is, 38 of what it takes 7 of to make a whole cornbread. Thus each person gets $\frac{38}{7}$ cornbreads.

However if we had just studied whole number division and had not yet been introduced to fractions, we might have simply divided 38 by 7 and concluded that

$$\begin{array}{rl} & \;\;5 \quad R3 \\ 7 & \overline{)\,38} \\ - & \;\;35 \\ \hline & \;\;\;3 \end{array}$$

Hence we would be able to give each person 5 cornbreads and we would still have 3 of what we needed 7 of in order to be able to give each person 1 more cornbread. From here it is a relatively easy step to get to the notion of $5\frac{3}{7}$.

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ery time you fill this cup 3 times, it is equivalent to 1 whole cup. Hence if you fill it 6 times you have 2 whole cups and then 1 more times will give you $2\frac{1}{3}$ cups. However, just as in pre-slicing cornbreads, there is nothing wrong with saying that you used 7 “thirds of a cup”; that is 7 of what it takes 3 of to equal 1 whole cup.

[4]Just as in the study of any foreign language, the sooner a student is introduced to a mathematical concept, the better the chances are for the student to become at ease with the concept. Waiting too long after the introduction of whole number division to introduce fractions gives the student the feeling that there is little if any connection between the two concepts. That is, they see fractions as cutting a pie into pieces of equal size and then taking a certain number of the pieces; but they do not see this as the same thing as dividing two whole numbers.

<u>Notes</u>:

- **If we had decided to use the above example to introduce the notation of common fractions, we might have had students see that the answer to $35 \div 7$ could be written as $\frac{35}{7}$ but that it is much more "natural" to rewrite the answer in the simpler but equivalent form, 5. In a similar way, rather than write that $42 \div 7 = \frac{42}{7}$, we write $42 \div 7 = 6$. However, suppose we are now given the problem $38 \div 7$. We can still write the answer as $\frac{38}{7}$ but in terms of writing the answer as a whole number, all we can say is that the quotient is greater than 5 (i.e., $5 \times 7 = 35$) but less than 6 (i.e., $6 \times 7 = 42$).**

- **A good way to see the connection between $5\frac{3}{7}$ and $\frac{38}{7}$ is to divide the cornbreads differently. For example suppose that a restaurant calls itself the "Lucky 7" because it slices each of its famous cornbreads into 7 equally sized servings. Then if 38 customers each order a serving of cornbread, they have ordered 38 of what it takes 7 of to make a whole cornbread. That is, they have ordered $\frac{38}{7}$ cornbreads. If we now divide 38 by 7 we see that the restaurant sold $5\frac{3}{7}$ cornbreads.**