Unedited Draft

Arithmetic Revisited

Lesson 3 :

The Arithmetic of Common Fractions

Appendix to Lesson 3

Some Elementary Number Theory

Part 8.3: Cartesian Products

1. The Cartesian Plane

The number line is a device whereby points on a line are named by numbers. An everyday model of the number line is an ordinary ruler. More specifically, we may visualize the ruler as being a"straight edge" on which points[1] are marked off at equally spaced intervals. These points are then named by numbers. Thus, for example, the point which is 2 units from the beginning of the ruler is then given the name 2. In essence the ruler is a marriage between geometry (points on a line) and arithmetic (numerical names for the points).

The number line is an extension of the ruler. Namely, we replace the ruler by a straight line that originates at a point which we denote by 0 and which is drawn in the direction from left to right. The concept of the number line dates back to the time of the ancient Greeks who visualized numbers geometrically as either points or lengths of line segments. They had no concept of negative numbers in the sense that it made no sense to them to think of a length that was so short that even if it had been 2 units longer would still be invisible!

However in the 17th century Rene Descartes showed that negative numbers were as "real" as the other numbers that the ancient Greeks had denoted by points on the number line. He simply extended the number line in the right-to-left direction through 0 and then used "positive (+)" to denote the left-to-right direction and "negative (–)" to denote the right-to-left direction. In that context the point that was 2 units to the right of 0 was denoted by $^{+}2$ and the point that was 2 units to the left of 0 was denoted by $^{-}2$.

[1]Actually there is subtlety involved here. Technically speaking a point has no thickness. So to see it we represent it as a dot. Conceptually there is a huge difference between a point (which is invisible) and a dot (which is visible), However, in everyday usage we usually refer to the dots as points.

Descartes actually went one step further by drawing a second number line that was vertical and also passed through the point 0. In this way he was able to identify any point in the plane with an ordered pair of numbers. For example, given a point P he could measure its horizontal distance from the vertical number line (which he called the y-axis) and its vertical distance from the horizontal line (which he called the x-axis). He listed the distance from the y-axis first (this distance was called the x-coordinate of the point) followed by the distance from the x-axis (this distance was called the y-coordinate of the point). In this way P(2,3) meant the point that was 2 units to the right of the y-axis and 3 units above the x-axis)[2].

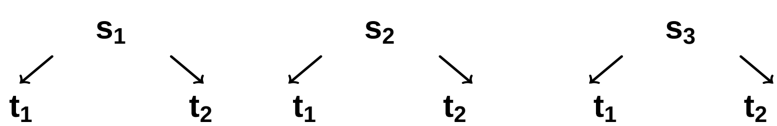
In deference to Descartes, we call this representation of points in the plane, the Cartesian Plane. It is the set of all ordered pairs where the first member of the pair denotes the distance of the point from the y-axis and the second member of the pair denotes the distance of the point from the x-axis.[3]

2. What Is a Cartesian Product?

The concept of the Cartesian Plane became generalized and was called the "Cartesian Product of Two Sets". For example, if S is any set that has 3 members and T is any set that has 2 members then the set of all ordered pairs in which one member of the pair comes from set S and the other from set T is called the Cartesian Product of S and T and is denoted by $S \otimes T$.[4]

With S and T as above, the number of ordered pairs is 3×2.

To see why this is true we may use a "tree diagram" as follows. Starting with any of the 3 members of S, there are 2 members of T with which it can be paired. In fact if we denote the 3 members of S by s_1, s_2 and s_3 and the 2 members of T by t_1 and t_2, the ordered pairs would be (s_1,t_1), (s_1,t_2), (s_2,t_1), (s_2,t_1), (s_3,t_1) and (s_3,t_2). We can illustrate this result with the following diagram..



[2]Notice the importance of order here. For example (3,2) is not the same as (3,2). More specifically (3,2) names the point that is 3 units to the right of the y-axis and 2 units above the x-axis.
[3]This can be a bit tricky in the sense that the x-coordinate is the distance from the y-axis and the y-cooridnate is the distance from the x-axis.
[4]Most mathematicians write the Cartesian Product of the two sets S and T as $S \times T$. However because this symbol looks very much like the same times sign we use when we multiply numbers, we will try to avoid confusion by using the symbol $\otimes$ to denote the Cartesian Product.

While the order of the pairs makes a difference (for example, (1,4) is not the same point as (4,1)), the number of ordered pairs remains the same. That is, if we write the member of T first, the ordered pairs would be (t_1, s_1), (t_1, s_2), (t_1, s_3), (t_2, s_1), (t_2, s_2) and (t_2, s_3). More pictorially:

$$\begin{array}{ccccccc} & t_1 & & & & t_2 & \\ \swarrow & \downarrow & \searrow & & \swarrow & \downarrow & \searrow \\ s_1 & s_2 & s_3 & & s_1 & s_2 & s_3 \end{array}$$

We may generalize this discussion in terms of the following definition:

***Definition*:**

The Cartesian Product of the two sets, S and T, is the set of all ordered pairs where the first member comes from set S and the second member comes from set T. We write the Cartesian Product set as $S \otimes T$.

***Important Point*:**

If we let N(S) denote the number of members in S; N(T), the number of members in T; and $N(S \otimes T)$, the number of members in their Cartesian Product (that is, the number of ordered pairs), then

$$N(S \otimes T) = N(S) \times N(T).$$

Illustrative Example 1:

How many 2-digits numbers can be formed from the digits 1, 2, 3, 4 and 5 if a number is allowed to contain the same digit twice?

***Solution*:**

In this case, we might let both S and T [5] denote the set whose members are 1, 2, 3, 4 and 5. Then there are as many ways of forming the 2-digit number as there are ways to pick one number from each of the sets S and T. In other words, we are looking for $N(S \otimes T)$ where $N(S) = N(T) = 5$. Using the result that $N(S \otimes T) = N(S) \times N(T)$, we see that $N(S \otimes T) = 5 \times 5$, or 25.

[5] Just as may denote variables in algebra by any symbols we wish to use, there is no need to use the letters S and T to denote the set of numbers 1, 2, 3, 4 and 5. We used S and T simply to parallel the letters we used in the definition of Cartesian Product. Also, because in this example S and T denote the same set, their Cartesian Product $S \otimes T$ could just as well be denoted $S \otimes S$.

Discussion:

To see how the formula works, we could think as follows:

The first number must be either 1, 2, 3, 4 or 5. Then no matter what the first number is, the second number must also be either 1, 2, 3, 4 or 5. In other words, if the first number is 1, the only two digit numbers we can form if 1 is in the tens place are 11, 12, 13, 14, and 156. We can repeat this process with the tens digit being 1, 2, 3, 4 or 5. Thus the possible numbers are:

11	12	13	14	15
21	22	23	24	25
31	32	33	34	35
41	42	43	44	45
51	52	53	54	55

We could also have let the first number denote the ones place. In this case our chart might look like:

11	21	31	41	51
12	22	32	42	52
13	23	33	43	53
14	24	34	44	54
15	25	35	45	55

Of course, if the sets S and T contain sufficiently many members it may be tedious to list all the combinations. For example:

Illustrative Example 2:

How many acronyms [6] can be made up if each acronym is supposed to contain exactly 2 letters of the English alphabet?

***Solution*:**

In this case, both S and T may be used to represent the set of all letters in the English alphabet. We are looking for the number of members in $S \otimes T$ and we simply use the formula $N(S \otimes T) = N(S) \times N(T)$, with $N(S) = N(T) = 26$. We then see that there are 26×26 or 676 such acronyms.

Discussion:

Nothing prevents us from specifically listing the acronyms to obtain AA, AB, AC, AD, etc., but by the time we listed all of the possibilities our list would contain 676 such acronyms. The key is that any of the 26 letters of the alphabet can be the first letter and once the first number

[6] Anacronym is a worfd compost\ed of the first letters of the words in a phrase For example US is an acronym for “United States”. and gcf is an acronym for “greaterst common factor”.

is chosen the second letter can also be any one of the 26 letters in the alphabet.

However, there are times when we have to be careful about what the members must look like. In short, reading comprehension is important.

For example:

<u>Illustrative Example 3:</u>

How many 2-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 if no number can contain the same digit twice?

<u>Solution:</u>

This time we can still choose the tens (or the ones) digit in 5 ways. That is, it can be either 1, 2, 3, 4 or 5. However, once that digit is chosen, it cannot be chosen again. Hence the second digit can be chosen in just 4 ways (namely any one of the digits that wasn't chosen as the first digit). In this case there are 5 ways to pick the first number and only 4 ways to pick the second. In other words, there are 5×4 or 20 different numbers that can be formed. More specifically, they are the numbers that are left from the 25 numbers we obtained previously after the double-digit numbers are eliminated. That is:

•	12	13	14	15
21	•	23	24	25
31	32	•	34	35
41	42	43	•	45
51	52	53	54	•

3. Cartesian Products of Three or More Sets

Just as there is an arithmetic for numbers there is also an arithmetic of sets. For example, just as whole number arithmetic obeys the associative property of multiplication, sets obey the associative property for Cartesian Products. In other words, we can show that if R, S and T denote any three sets, $(R \otimes S) \otimes T = R \otimes (S \otimes T)$.[7]. For example, suppose a small restaurant has 5 different kinds of sandwiches, 4 different kinds of beverages and 3 different kinds of desserts and that a meal consists of one sandwich, one beverage and one dessert. We want to know how many different meals the restaurant can serve.

-- We will let S denote the set of sandwiches; B, the set of beverages; and D, the set of desserts.

-- Clearly, in choosing a meal, it makes no difference whether we order a sandwich, beverage or a dessert first. Thus, for example, we might first order our sandwich and beverage and then order the dessert. That is, we might first order a member of $S \otimes B$ followed by a member of D. In other words, we have chosen a meal from the set $(S \otimes B) \otimes D$.

-- On the other hand we could have ordered the same meal by first ordering a sandwich (that is, choosing a member of S) and then add to it a beverage and a dessert (that is, a member of $B \otimes D$). In other words, the same meal is a member of the set $S \otimes (B \otimes D)$.

***In summary*:**

If A, B and C are any three sets, $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

Note:

For this reason we can write $A \otimes B \otimes C$ without worrying about where to place the grouping symbols.

The same reasoning that allowed us to conclude that the Cartesian Product was associative allows us to conclude that

$$N(A \otimes B \otimes C) = N(A) \times N(B) \times N(C).$$

Namely, we can pick the components of the "triplet" (a, b, c) in any order that we wish.

[7] Keep in mind that the members of $S \otimes T$ are ordered pairs. In other words, if s is a member of S and t is a member of T, then the corresponding member of $S \otimes T$ is (s, t)

Illustrative Example 4:

In how many ways can we form a 3-digit numeral using the digits 1, 2, 3, 4, and 5 if no numeral can contain the same digit more than once?

***Solution*:**

We can let N(A) denote the number of ways in which the first digit can be chosen (and we can assume that this will be the hundreds digit but it doesn't really matter). Since we may choose any one of the five digits, N(A) $=$ 5. We can then let N(B) denote the number of ways in which we can choose the next digit. Since there were five digits from which to choose the first digit and no digit can be repeated, there are only four ways in which to choose the second digit. That is, N(B) $=$ 4. If we let N(C) denote the number of ways in which we can choose the next digit, notice that since two of the five digits have already been used and no digit can be repeated, N(C) $=$ 3. Hence the total number of 3-digit numerals, N(A $\otimes$ B $\otimes$ C), that can be formed under the given conditions is N(A) $\times$ N(B) $\times$ N(C) $=$ 5 $\times$ 4 $\times$ 3 = 60.

Discussion:

In the previous example we had already listed the twenty 2-digit numerals that could be formed from the digits 1, 2, 3, 4 and 5 if no numeral contained a repeated digit. All we are now saying is that each of these twenty possibilities leads to three 3-digit numerals. For example, if we already had obtained 34, the third digit had to be either 1, 2 or 5. That is, we could augment the 2-digit numeral 34 in three ways to form the 3-digit numerals 341, 342, or 345. If we wished to, as shown below, we could list the entire 60 possibilities but as the number of members of the sets increases, the listing process becomes rather tedious!

123	213	312	412	512
124	214	314	413	513
125	215	315	415	514
132	231	321	421	521
134	234	324	423	523
135	235	325	425	524
142	241	341	431	531
143	243	342	432	532
145	245	345	435	534
152	251	351	451	541
153	253	352	452	542
154	254	354	453	543

4. Cartesian Products and Finding Divisors of Whole Numbers

The concept of the Cartesian Product plays a big role in problems in which we want to count [8] the number of ways in which one or more events can occur. For now, however, we want to demonstrate how they can be used, in conjunction with prime factorization, to determine the total number of factors a number has. The approach is based on the fact that every natural number can be written uniquely as a product of prime numbers.

To see how this works, let's revisit a previous example. More specifically, let's look at how we can count the factors of 36.

-- We have already seen that 36 $= 2^2 \times 3^2$. Because 2 is a prime number, the only factors of 2^2 are 1 (2^0), 2^1 and 2^2. So let A be the set whose members are 1, 2 and 4. Similarly, since 3 is also a prime number, the only factors of 3^2 are 3^0, 3^1 and 3^2. So let B denote the set whose members are 1, 3 and 9. Every factor of 36 is therefore composed of a product of two numbers, one of which comes from A and the other from B.[9] Thus the total number of ordered pairs whose product is 36 is equal to N(A $\otimes$ B); which in turn is given by N(A) $\times$ N(B), which is 3 $\times$ 3 or 9.

-- The only remaining question is whether two different ordered pairs can have the same product. In other words, for any member (a_1, b_1) in A $\otimes$ B we can form the product a_1b_1. The question then becomes:

Is there any other member (x, y) of A $\otimes$ B for which $xy = a_1b_1$?

Because of the Prime Factorization Theorem, the answer is "no!" Rather than give a rigorous proof let's simply demonstrate it with a specific example. We know that 18 is a factor of 36 and that 18 can be written as $2^1 \times 3^2$. By the Prime Factorization Theorem no other combination of powers of 2 and 3 can have 18 as its product.

-- Thus every member (a, b) of A $\otimes$ B "codes" a *different* factor of 36 [10]. In other words, we can identify each ordered pair (a,b) in A $\otimes$ B with the product a $\times$ b and no two of these products can be equal. This is illustrated in the "graph" below:

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[8]We emphasize the word "count". For example, if a restaurant has 25 different kinds of sandwiches, 15 different kinds of beverages and 20 different desserts, we can use the concept of the Cartesian Product to conclude that there are $25 \times 15 \times 20$ or 7,500 different meals that one could order if a meal consisted of one sandwich, one beverage and one dessert. This is quite different from actually having to list all possibilities and then counting them.

[9]Notice that these combinations include those numbers that belong only to A and/or only to B. That is, since 1 is a member of both sets, we can write the members of A as 1×1, 1×2 and 1×4; and we can write the members of B as 1×1, 1×3 and 1×9.

[10]More specifically, the "code" is defined by assigning the value ab to (a,b)

4	$(1,4)$ or 1×4 (4)	$(3,4)$ or 3×4 (12)	$(9,4)$ or 9×4 (36)
2	$(1,2)$ or 1×2 (2)	$(3,2)$ or 3×2 (6)	$(9,2)$ or 9×2 (18)
1	$(1,1)$ or 1×1 (1)	$(3,1)$ or 3×1 (3)	$(9,1)$ or 9×1 (9)
	1	3	9

Note: In the above grid, the ordered pair (a, b) stands for $a \times b$ [11].

Illustrative Example 5:

How many factors does the number N have if $N = 2^3 \times 3^2 \times 5$?

***Solution*:**

-- The only factors of 2^3 are 1, 2^1, 2^2 and 2^3. So let A be the set whose members are 1, 2, 4 and 8. Hence $N(A) = 4$.

-- The only factors of 3^2 are 1, 3^1 and 3^2. So let B be the set whose members are 1, 3 and 9. Hence $N(B) = 3$.

-- The only factors of 5 are 1 and 5. So let C be the set whose members are 1 and 5. Hence $N(C) = 2$.

-- Every factor of N consists of a member from each of the sets A, B and C. Therefore the number of factors of N must be $N(A \otimes B \otimes C)$ which, in turn is given by $N(A) \times N(B) \times N(C)$ or $4 \times 3 \times 2$. In other words, N has 24 factors; and each factor has the form $a \times b \times c$, where a is a member of A; b is a member of B; and c is a member of C..

[11] We have used the type of grid that is usually found on a map rather than a replica of the $xy-$ plane in order to best highlight the process we are using.

<u>Discussion</u>:

Again we can use a graph to list the 24 factors. We can begin, for example, by listing the products we obtain when we multiply each member in A by a member in B. That is:

8	(1, 8) **or** 1×8 **(8)**	(3, 8) **or** 3×8 **(24)**	(9, 8) **or** 9×8 **(72)**
4	(1, 4) **or** 1×4 **(4)**	(3, 4) **or** 3×4 **(12)**	(9, 4) **or** 9×4 **(36)**
2	(1, 2) **or** 1×2 **(2)**	(3, 2) **or** 3×2 **(6)**	(9, 2) **or** 9×2 **(18)**
1	(1, 1) **or** 1×1 **(1)**	(3, 1) **or** 3×1 **(3)**	(9, 1) **or** 9×1 **(9)**
	1	**3**	**9**

Thus the 12 factors of N that have only 2 and 3 as prime divisors has as its members 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36 and 72. Each of these twelve factors can then be multiplied by either 1 or 5 to obtain all the factors of N. More specifically:

5	(1, 5 or 1×5 **(5)**	(2, 5) or 2×5 **(10)**	(3, 5) or 3×5 **(15)**	(4, 5) or 4×5 **(20)**	6, 5) or 6×5 **(30)**	(8, 5) or 8×5 **(40)**	(9, 5) or 9×5 **(45)**	(12, 5) or 12×5 **(60)**	(18, 5) or 18×5 **(90)**	(24, 5) or 24×5 **(120)**	(36, 5) or 36×5 **(180)**	(72, 5) or 72×5 **(360)**
1	(1, 1) or 1×1 **(1)**	(2, 1) or 2×1 **(2)**	(3, 1) or 3×1 **(3)**	(4, 1) or 4×1 **(4)**	(6, 1) or 6×1 **(6)**	(8, 1) or 8×1 **(8)**	(9,1) or 9×1 **(9)**	(12.1) or 12×1 **(12)**	(18, 1) or 18×1 **(18)**	(24, 1) or 24×1 **(24)**	(36, 1) or 36×1 **(36)**	(72, 1) or 72×1 **(72)**
	1	2	3	4	6	8	9	12	18	24	36	72

***<u>Note</u>*: The bottom row of the chart is just a repetition of the 12 factors of $2^3 \times 3^2$ and the top row of the chart represents the factors we obtain by multiplying each number on the bottom row by 5.**

The following chart may illustrate the above procedure in greater detail. We have made a portion of a chart in which each column represents the powers of a different prime number. In our chart we have made columns for the first eight prime numbers and each column contains the powers through the 5th.

I	II	III	IV	V	VI	VII	VIII
1	1	1	1	1	1	1	1
2^1	3^1	5^1	7^1	11^1	13^1	17^1	19^1
2^2	3^2	5^2	7^2	11^2	13^2	17^2	19^2
2^3	3^3	5^3	7^3	11^3	13^3	17^3	19^3
2^4	3^4	5^4	7^4	11^4	13^4	17^4	19^4
2^5	3^5	5^5	7^5	11^5	13^5	17^5	19^5

The basic idea is that every natural number (greater than 1) can be written in one and only one way as a product in which no more than one entry from each column appears.

5. An Historic Note About Divisors

The fact that there are 60 minutes in an hour means that many fractional parts of an hour will be whole number of minutes. This type of reasoning helps to explain why there are 5,280 feet in one mile, even though 5,000 would have been an easier number to memorize. The answer lies in the fact that 5,280 has more factors than does 5,000; and this in turn means that many fractional parts of a mile will be a whole number of feet.

So, if only for practice, let's count the number of factors (divisors) 5,280 has. We begin by breaking down 5,280 into a product of prime numbers. This can be done in several ways but we will pick a rather straight forward way, looking for prime factors. For example, 5,280 ends in 0 and this tells us that it is divisible by 2 [12]. In fact:

$$5{,}280 = 2 \times 2{,}640|$$

2,640 also ends in a 0 so it too is divisible by 2. In fact $2{,}640 = 2 \times 1{,}320$. Hence:

$$5{,}280 = 2 \times 2 \times 1{,}320$$

Since 1,320 also ends in a 0 it is also divisible by 2. In fact $1{,}320 = 2 \times 660$. Therefore

$$5{,}280 = 2 \times 2 \times 2 \times 660$$

Continuing this process we see that $660 = 2 \times 330$, Thus:

[12] It also tells us that 5,280 is divisible by 5. However, no matter what order we proceed in, the prime factorization of 5,280 will be the same,

$$5{,}280 = 2 \times 2 \times 2 \times 2 \times 330$$

And 330 = 2 × 165. Hence

$$5{,}280 = 2 \times 2 \times 2 \times 2 \times 2 \times 165$$

165 ends in a 5, hence it is divisible by 5. In fact, 165 = 5 × 33. Therefore:

$$5{,}280 = 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 33$$

And since 33 = 3 × 11, we see that

$$5{,}280 = 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 3 \times 11$$

And since 11 is a prime number the prime factorization is complete. Written in exponential form, we see that

$$5{,}280 = 2^5 \times 3 \times 5 \times 11$$

The divisors of 2^5 are 1, 2, 2^2, 2^3, 2^4 and 2^5. Hence there are 6 divisors of 2^5. These are the divisors of 5,280 that are not divisible by 3, 5 or 11,

The divisors of 3 are 1 and 3. These are the 2 divisors of 5,280 that are not divisible by 2, 5 or 11.

The divisors of 5 are 1 and 5. These are the 2 divisors of 5,280 that are not divisible by 2, 3 or 11.

The divisors of 11 are 1. These are the 2 divisors of 5,280 that are not divisible by 2, 3 or 5.

Hence all in all, 5,280 has 6 × 2 × 2 × 2 or 48 divisors.[13]

> **<u>Note</u>:**
>
> **By breaking a given number into its prime factorization, we can immediately tell whether it is a divisor of 5,280. More specifically, the fact that 5,280 = 2 × 2 × 2 × 2 × 2 × 5 × 3 × 11 means that any divisor of 5,280 has a to be a combination of these prime factors. So, for example, 14 breaks down into 2 × 7; but since 7 is not among the prime factors of 5,2280, 14 cannot be a divisor of 5,280. On the other hand, since 15 = 5 × 3 and both 3 and 5 are prime factors of 5,280 we know that 15 is a divisor of 5,280. In fact, simply by rewriting the factors of 5,280 in the form (5 × 3) × 2 × 2 × 2 × 2 × 2 × 11 = 15 × 352, we see that 5,280 ÷ 15 = 352.**

[13]On the other hand 5,000 = 5 × 1,000 = 2 × 5 × 2 × 5 × 2 × 5 = $2^3 \times 5^4$. 2^3 has 4 divisors (1, 2, 4 and 8) and 5^4 has 5 divisors (1, 5, 25, 125, 625 and 3,125). Therefore 5,000 only has 4 × 5 or 20 divisors.

If we wanted to, we could use the method shown in a previous illustration to list the 48 divisors of 5,280. Namely we could begin by listing the members of A ⊗ B where A = {1, 2, 4, 8, 16, 32} and B = {1,3} to obtain:

3	(1, 3) or 1 × 3 (3)	(2, 3) or 2 × 3 (6)	(4, 3) or 4 × 3 (12)	(8, 3) or 8 × 3 (24)	(16, 3) or 16 × 31 (48)	(32, 3) or 8 × 1 (96)
1	(1, 1) or 1 × 1 (1)	(2, 1) or 2 × 1 (2)	(4, 1) or 4 × 1 (4)	(8, 1) or 8 × 1 (8)	(16, 1) or 16 × 1 (16)	(32, 1) or 32 × 1 (32)
	1	2	4	8	16	32

We may then let C denote A ⊗ B and from the above chart we see that the 12 factors of 5,280 that are represented by C are 1, 2 3, 4, 6, 8, 12, 16, 24, 32, 48, and 96. These are the 12 divisors of 5,280 that are not divisible by either 5 or 11.

We can then form the Cartesian Product of C and D where D = {1,5} to obtain:

5	(1, 5 or 1 × 5 (5)	(2, 5) or 2 × 5 (10)	(3, 5) or 3 × 5 (15)	(4, 5) or 4 × 5 (20)	6, 5) or 6 × 5 (30)	(8, 5) or 8 × 5 (40)	(12, 5) or 12 × 5 (60)	(16, 5) or 16 × 5 (80)	(24, 5) or 24 × 5 (120)	(32, 5) or 32 × 5 (160)	(48, 5) or 48 × 5 (240)	(96, 5) or 96 × 5 (480)
1	(1, 1) or 1 × 1 (1)	(2, 1) or 2 × 1 (2)	(3, 1) or 3 × 1 (3)	(4, 1) or 4 × 1 (4)	(6, 1) or 6 × 1 (6)	(8, 1) or 8 × 1 (8)	(12,1) or 12 × 1 (12)	(16.1) or 16 × 1 (16)	(24, 1) or 24 × 1 (24)	(32, 1) or 32 × 1 (32)	(48, 1) or 48 × 1 (48)	(96, 1) or 96 × 1 (96)
	1	2	3	4	6	8	12	16	24	32	48	96

We may then let E denote D ⊗ E and from the above chart we see that the 24 factors of 5,280 that are represented by E are 1, 2 3, 4, 5, 6, 8, 12, 16, 24, 32, 48, and 96. More specifically these are the divisors of 5,280 that are not divisible by 11. We can see from the above chart that these 24 numbers are:

> **1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, 40, 48, 60, 80, 96, 120, 160, 240, 480**

The other 24 divisors of 5,280 can be found by multiplying each of the above 24 divisors of 5,280 by 11 to obtain:

11, 22, 33, 44, 55, 66, 88, 110, 132, 165, 176, 220, 264, 330, 352, 440, 528, 660, 880, 1,056, 1,320, 1,760, 2,640, 5,280.

Combining these 24 divisors of 5,280 with the previous 24 divisors of 5,280 we see that the complete list of divisors of 5,280 are:

1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 15, 16, 20, 22, 24, 30, 32, 33, 40, 44, 48, 55, 60, 66, 80, 88, 96, 110, 120, 132, 160, 165, 176, 220, 240, 264, 330, 352, 440, 480, 528, 660, 880, 1,056, 1,320, 1,760, 2,640 and 5,280.

The important point to observe here is that by choosing 5,280 to be the number of feet in a mile, many fractional parts of a mile are a whole number of feet.

6. Using Prime Factorization To Find LCM's and GCF's

It's not always convenient to decompose a number into a product of prime numbers. However, when it is convenient, it gives us a rather nice way to compute the gcd and/or the lcm of a group of numbers. In order to "cut to the chase" as quickly as possible, let's look at three numbers that are already written as products of primes.

Illustrative Example #6:

What is the greatest common factor of a, b, and c if $a = 2 \times 3^2 \times 5$, $b = 2^2 \times 3 \times 5^3$ and $c = 2 \times 3^4$?

***Answer*:**

$\text{gcf}(a, b, c) = 2 \times 3 = 6$

Solution:

The gcf has to be a factor of each of the numbers. Hence there is no need to consider any prime factors other that 2, 3 and 5. 2 is a factor of each number, hence one of the factors of the gcf will be 2. 2^2 is a factor of b but to be a common factor it has to be a factor of all the numbers. Hence 2 is the greatest power of 2 that is a common factor of a, b and c. 3 is a factor of all three numbers but 3^2 is not a factor of b. Hence the greatest power of 3 that is a common factor of all three numbers is 3. 5 is a factor of both a and b but not of a. Hence 5 is not a common factor of a, b and c. Therefore the greatest common factor of a, b and c is 2×3.

Discussion:

In more "traditional" language, $a = 90$, $b = 1{,}500$ and $c = 162$. What we have done in this problem is to show that $\text{gcf}(90, 1{,}500, 162) = 6$. In particular:

$$\begin{aligned} 90 &= 6 \times 15 \\ 1{,}500 &= 6 \times 250 \\ 162 &= 6 \times 27 \end{aligned}$$

Notice that the numbers we get upon dividing 90, 1,500 and 162 by 6 form a set of relatively prime numbers; namely 15, 250 and 27 [14]. This must be the case. Namely, if we assume that d is a common factor of 15, 250 and 27 then $6 \times d$ would also be a factor of these three numbers. However, since we already know that the great common factor is 6, d must equal 1.

The process used in the above example can be generalized. Namely to find the greatest common divisor of a set of numbers, we write each of the numbers as a product of prime numbers and we then "factor out" the least power of each of each of the prime factor. The resulting product is the greatest common factor.

Illustrative Example #7:

What is the greatest common factor of 72, 144 and 432?

***Answer*:**

gcf (72, 432,144) = 72

***Solution*:**

Using prime factorization we see that:

$$\begin{aligned} 72 &= 2^3 \times 3^2 \\ 144 &= 2^4 \times 3^2 \\ 432 &= 2^4 \times 3^3 \end{aligned}$$

We know that the only prime numbers that can be common factors of all three numbers are 2 and 3.

-- In other words, the greatest common factor of 72, 432 and 144 must have the form $2^n \times 3^m$ where n and m are whole numbers.

-- We see that the least number of times 2 appears as a factor in the given numbers is 3. Hence we know that 2^3 is one factor of the gcf.

-- We next observe that the least number of times 3 appears as a factor in the given numbers is 2. Hence we know that 3^2 is the other factor of the gcf.

-- In summary, then gcf (72, 432, 144) $= 2^3 \times 3^2 = 8 \times 9 = 72$

To find the least common multiple of a set of numbers, we "extract" the greatest power of each prime factor rather than the least power.

[14] Notice that 15 and 250 have 5 as a common factor and that 15 and 27 have 3 as a common factor. However neither 3 nor 5 can be a common factor because 5 is not a factor of 27 and 3 is not a factor of 250.

Illustrative Example #8:

What is the least common multiple of 30, 45 and 144?

***Answer*:**

lcm (30, 45, 144) = 720

***Solution*:**

Using prime factorization we see that:

$$\begin{aligned} 30 &= 2 \times 3 \times 5 \\ 45 &= 3^2 \times 5 \\ 144 &= 2^4 \times 3^2 \end{aligned}$$

-- Since the given numbers have only 2, 3 and/or 5 as their prime factors, the least common multiple has no need to contain other prime factors.

-- The greatest number of factors of 2 that appear in any one of the numbers is 4 (namely, in the prime factorization of 144). Hence there is no need to use a greater power of 2 in our search for the lcm.[15]

-- The greatest number of times 3 appears as a factor in any of the numbers is 2 (namely in the prime factorizations of 45 and 72). Hence there is no need to use a greater power of 3 in our search for the lcm.

-- The factor 5 never appears more than once in any of the numbers. Hence there is no need to use more than one factor of 5 in the lcm.

-- Putting the above observations together we see that the least common multiple of 30, 45 and 144 is

$$2^4 \times 3^2 \times 5^1 = 16 \times 99 \times 5 = 720.$$

As a final application of finding the lcm of a group of numbers, we will look at an addition problem involving fractions.

[15]That is, 2^4 is divisible by each of the numbers f 1, 2^1, 2^2 and 2^3.

Illustrative Example #9:

Express the sum $\frac{7}{30} + \frac{8}{45} + \frac{5}{144}$ as a common fraction in lowest terms.

Answer: $\frac{107}{240}$

Solution:

We know that:

$$\begin{aligned} 720 &= 30 \times 24 \\ &= 45 \times 16 \\ &= 144 \times 5 \end{aligned}$$

Hence we may rewrite the problem as:

$$\begin{aligned} \frac{7}{30} + \frac{8}{45} + \frac{5}{144} &= \frac{7 \times 24}{30 \times 24} + \frac{8 \times 16}{45 \times 16} + \frac{5 \times 5}{144 \times 5} \\ &= \frac{168}{720} + \frac{128}{720} + \frac{25}{720} \\ &= \frac{321}{720} \\ &= \frac{107}{240} \end{aligned}$$

Note:

To reduce $\frac{321}{720}$ to lowest terms, it is not necessary to write 321 as a product of prime numbers (although it's certainly permissible to do so!). The point is that we already know that the prime factors of 720 are 2, 3 and 5. Because we can only cancel common factors, we need only check whether 321 is divisible by either 2, 3 or 5. No other prime factors matter!

We can quickly check that 321 is not divisible by either 5 or 2 but it is divisible by 3. Thus we may rewrite $\frac{321}{720}$ as $\frac{3 \times 107}{3 \times 240}$, from which we can conclude that $\frac{321}{720}$ is equal to $\frac{107}{240}$. Finally since the only prime factors of 240 are 2, 3 and 5; and since neither 2, 3 nor 5 are factors of 107, we see that $\frac{107}{240}$ is already in lowest terms.[16]

This concludes our present discussion of the role of prime numbers in the study of arithmetic.

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[16] Of course, we might have recognized in this example that 107 is a prime number and conclude that the fraction was therefore in lowest terms but the point is that the method we've just described works in every case.