Unedited Draft

Arithmetic Revisited

Lesson 2:

The Role of Place Value in the Development of Whole Number Arithmetic

Part 5: An Introduction to Number Bases

Enrichment

1. A Personal Preface

Prior to the "New Math", little attention was paid to the concept of different number bases. During the "New Math" era there was much attention given to different number bases but, for whatever reason, the end result was that in most places the study degenerated into a series of rote exercises. Maybe because of this, the study of different number bases has now been minimized or even dropped from most elementary school math curricula.

It is our hope that this lesson will provide insight into the structure of place value arithmetic as well as to show the important distinction between numbers and how we elect to represent the numbers. Moreover, having students do problems in different number bases and then checking the correctness of their computations by translating the problems back into the more familiar base-ten system provides an almost painless drill session.[1] With the proper choice of examples, students curiosity can be expanded and their desire to work with numbers (actually, numerals) is often enhanced.

To begin with, the concept of place value is independent of the number ten. All that is necessary is that we always "trade in" the same number of one denomination for one of the next greater denomination [2]. It is our opinion that one problem in emphasizing base ten arithmetic is that there are a lot of "number facts" (i.e., the addition and multiplication tables) to internalize and this can obscure the true meaning of place value.

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[1]And from a teacher's point of view, trying to do arithmetic in a base other than ten helps teachers appreciate the difficulties primary grade students encounter when they try to learn base ten arithmetic.

[2]In fact prior to the invention of place value twelve, rather than ten, was often sued as a base. For example there were 12 in a dozen and a gross was 12 twelve's. Twelve was chosen over ten because it had more proper divisors. That is the proper divisors of 10 are only 2 and 5; while the proper divisors of 12 are 2, 3, 4 and 6.

For example, suppose we had decided to trade in by two's rather than be tens. In that case, we would have what might be called "The School Children's Delight"; namely the only tables the student would have to know would be the 0 and 1 tables. More specifically, whenever there were two of any denomination we could trade them in for one of the next greater denomination.

Since most of us can internalize numbers when money is involved, we could play the game of "Let's Pretend" and imagine that we have invented a monetary system in which two $1-bills could be traded in for one $2-bill; two $2-bills could be traded in for one $4-bill; two $4-bills could be traded in for one $8-bill etc. We could then represent any whole number of dollars by a sum that contained no more than one of any denomination. For example:

$8	$4	$2	$1	
			1	→ $1
		1		→ $2
		1	1	→ $3
	1			→ $4
	1		1	→ $5
	1	1		→ $6
	1	1	1	→ $7
1				→ $8
1			1	→ $9

We can begin to transform the above table into place value format by using 0's as place holders whenever needed. That is:

$8	$4	$2	$1	
			1	→ $1
		1	0	→ $2
		1	1	→ $3
	1	0	0	→ $4
	1	0	1	→ $5
	1	1	0	→ $6
	1	1	1	→ $7
1	0	0	0	→ $8
1	0	0	1	→ $9
1	0	1	0	→ $10
1	0	1	1	→ $11

Thus in base two arithmetic, rather than learning to count:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

children would learn instead:

1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011;

and the arithmetic tables would be:

$+$	0	1
0	0	1
1	1	10

$\times$	0	1
0	0	0
1	0	1

At the kindergarten level, it's possible for children to learn a bit about base two arithmetic at the same time they are learning, say, the colors of the rainbow. For example, rather than using money as an illustration, they could use tiles that had the rainbow colors. Thus rather than exchanging two $1-bills for one $2-bill, etc., they could exchange 2 red tiles for 1 orange tile; 2 orange tiles for 1 yellow tile; 2 yellow tiles for 1 green tile, 2 green tiles for 1 blue tile; and 2 blue tiles for 1 purple tile.

By the first grade the tiles could be replaced by numbers; and the students could then use a “revisionist” monetary system to “graduate" to base three arithmetic and beyond.

2. Reliving Arithmetic Through the Eyes of Base-Two

By now we should be beginning to see how learning arithmetic may depend on the base we use for place value. One base might be more helpful than another in one situation but less helpful in another. In that respect, think about how often students complain about how difficult it is for them to "memorize" the multiplication tables. Think of how much easier they would find it to be if there were fewer than ten single digits to memorize. What, then, is the smallest base we could use?

The fact that in any number base the base is represented by the numeral 10 means that any number base must contain at least the two digits 0 and 1. Hence, the smallest number base that can exist is base two. It might not seem that a number base that small would be very practical; but, in fact, it is.

***Reason #1*:**

The fact that we have ten fingers made it rather natural for us to choose ten as the base of our place value system of numerals. However, computers, in a manner of speaking, have only two fingers. Namely, a switch is either off or on. Thus it is natural to use only 0 and 1 to "code" these two possibilities.

Reason #2:

The fact that a switch is either on or off means that we can use, say, 0 to stand for "off" and 1 to stand for "on". The point is that the same numeral scheme can be used in any situation in which there are only two mutually exclusive possibilities. Thus:

- **With respect to sets, an element is either in the set or it's not. In this case, it is traditional to let 0 represent "the element doesn't belong to the set" and 1 to represent "the element does belong to the set.**

- **With respect to flipping a coin, each toss will be either a head or a tail. We might, therefore, let 0 denote "tails" and 1 denote "heads".**

- **With respect to "true/false" statements, we might let 0 denote a false statement and 1 to denote a true statement.**

In any event, let's learn how the arithmetic would have looked if we had used base two instead of base ten. Practice Problem #1 serves as a review of some of our earlier comments.

Practice Problem #1:

In base-two arithmetic the only single digits are 0 and 1. Pretend you are a youngster who has just learned to write the numerals that represent the numbers from one through ten. Show how you would write these numbers using base-two numerals.

Answer: **1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1002**

Solution:

We could view a base-two odometer as our model for the number line. namely, suppose we have an odometer where each gear has only the two faces, 0 and 1. Whenever a face reads 1, after the next mile it would read 0 and make the face of the gear to its immediate left increase by 1. That is:

Miles Driven	Base-Two Reading	Base-Ten Reading
none	00000	0000
one	00001	0001
two	00010	0002
three	00011	0003
four	00100	0004
five	00101	0005
six	00110	0006
seven	00111	0007
eight	01000	0008
nine	01001	0009
ten	01010	0010
eleven	01011	0011
twelve	01100	0012
thirteen	01101	0013
fourteen	01110	0014
fifteen	01111	0015
sixteen	10000	0016

So just as we learned to memorize how numbers were written, a person living in the base-two system would learn to count 1, 10[3], 11, 100, 101, 110, 111, 1000, 1001, 1010 etc.

[3] To avoid confusing $(10)_{two}$ with what we call 10, read $(10)_{two}$ as if it were written $(\text{one-zero})_{two}$ In a similar way we'll read $(11)_{two}$ as $(\text{one-one})_{two}$, etc.

From the way counting proceeds in base two it's easy to see that the only arithmetic tables you would need in order to perform the arithmetic algorithms are:

Addition
0 + 0 = 0
0 + 1 = 1
1 + 0 = 1
1 + 1 = 10

Multiplication
$0 \times 0 = 0$
$0 \times 1 = 0$
$1 \times 0 = 0$
$1 \times 1 = 1$

While the tables are simple to learn, notice how many digits are necessary for us to use in order to express relatively small numbers. [4]

Practice Problem #2:

Using the above table find the sum of 111 and 101. In other words:

$$(111)_{two} + (101)_{two} = (\ ?\ )_{two}$$

Answer: $(111)_{two} + (101)_{two} = (1100)_{two}$

***A "Pre-Solution" Note*:**

One of the reasons that different number bases were eliminated from many curricula was that the topic simply became one of rote. That is, students would often simply translate a problem involving a different number base, as in this example, base-two, into the more familiar base-ten system; answer the question as a base-ten question and then translate the answer back into the different base.[5]

[4]For example, using base-ten numerals we see that 2^{20} = 1,048,576. However in the language of base-two arithmetic, we represent 2 as 10 and thus written in base ten 2^{20} would be represented by a 1 followed by twenty 0's. In other words, $(1{,}048{,}576)_{ten} = (100{,}000{,}000{,}000{,}000{,}000{,}000)_{two}$; which means a number that can be expressed using seven digits in base ten might require twenty one digits to express it in base two.

[5]This is analogous to how people first learn to answer a question that is posed in a foreign language that they are learning. That is, they hear the problem in the foreign language, silently translate it into their own language; still silently, they answer the question in their own language; and finally, they translate their answer back into the foreign language (Bilingual is when the person goes directly from hearing the problem in the foreign language to answering it in the foreign language). In this context, students never learned to understand different number bases in a "bilingual format"

Thus one way to solve this problem is to rewrite $(111)_{two}$ as 7 in base-ten and .$(101)_{two}$ as 5 in base-ten. So in the language of base-ten, the problem asks "Find the sum of 7 and 5". In the comfort of base-ten we see that the answer is 12. As our final step we rewrite 12 as it would be written in base-two namely;

$$12 = 1(8) + 1(4) + 0(2) + 0(1) = (1100)_{two}.$$

While this gives us the correct answer, it makes it seem as if we first had to know base-ten arithmetic in order to do base-two arithmetic.

However, in this and in the following problems we want to emphasize that the concept of place value is independent of the number base. In particular, no matter what base people were used to using, they would learn to count; after which they would learn their "tables"; and then use the same algorithms for arithmetic that we use in our own base-ten system. This is indicated in the solution below.

Solution:

-- We begin by writing the problem in exactly the same way we would have if this had been a base-ten problem.

$$\begin{array}{r} 1\ 1\ 1 \\ +\quad 1\ 0\ 1 \\ \hline \end{array}$$

-- Then starting in the ones place we begin the addition by looking at the table and saying "$1 + 1 = 10$. Put down the 0 and carry the 1".

$$\begin{array}{r} 1\ \ \ \\ 1\ 1\ 1 \\ +\quad 1\ 0\ 1 \\ \hline 0 \end{array}$$

-- We then move to the 10's (that is 10_{two}) place and we say "$1 + 1 = 10, 10 + 0 = 10$. Put down the 0 and carry the 1".

$$\begin{array}{r} 1\ 1\ \ \ \\ 1\ 1\ 11 \\ +\quad 1\ 0\ 1 \\ \hline 0\ 0 \end{array}$$

-- We then move to the 100's place and say "$1 + 1 = 10. 10 + 1 = 11$. Put down the 1 and carry the 1.

$$\begin{array}{r} 1\ 1\ 1\ \ \ \\ 1\ 1\ 1 \\ +\quad 1\ 0\ 1 \\ \hline 1\ 0\ 0 \end{array}$$

-- Finally, we move to the last place and bring down the 1, thus yielding:

	1	1	1	
		1	1	1
+		1	0	1
	1	1	0	0

Notes:

• The person doing arithmetic in base two would never think of writing $1+1=2$. Namely, the digit 2 doesn't exist in base two arithmetic. In essence the base two person memorizes how to count, learns the tables and then carries out the addition algorithm in exactly the same way that we do addition in our own base-ten system.

• As teachers, trying to think in terms of base two arithmetic might give you some indication of what youngsters go through when they first try to learn base ten arithmetic.

• Again, if it's easier to think in terms of money, we may view the problem in the form:

	$8	$4	$2	$1	
		1	1	1	
+		1	0	1	
		2	1	2	
		2	2	0	← We trade in two $1's for one more $2
		3	0	0	← We trade in two $2's for one more $4
	1	1	0	0	← We trade in two of our $4's for one $8.

While this is a nice way to visualize what we did more abstractly above, it again gives the illusion that we have to use the digits 2, 4 and 8 when we talk about base-two arithmetic.

- **Independently of any other uses for teaching different number bases, the topic gives us a good way to have students provide drill that otherwise might have seemed tedious.**

For example, given the number fact

$$(111)_{two} + (101)_{two} = (11000)_{two}$$

we can ask students to convert the numbers from base two to base ten and then see if the resulting statement is true. In terms of the present problem, we would see that:

	8	4	2	1		base ten
		1	1	1	→	7
+		1	0	1	→	5
	1	1	0	0	→	12

<u>Practice Problem #3</u>

Using the above table compute the value of $(1100)_{two} - (11)_{two}$. In other words, find the value of ?, if $(11)_{two} + (?)_{two} = (1100\ )_{two}$.

Answer: **$(1100)_{two} - (11)_{two} = (1001)_{two}$**

Solution:

<u>Method #1: The "Unadding" Technique</u>:

We're in the base-two system and we want to see what we must add to 11 to obtain 1100 as the sum. So we simply count from 11 to 1100. That is:

11,	100,	101,	110,	111,	1000,	1001,	1010,	1011,	1100
↑	↑	↑	↑	↑	↑	↑	↑	↑	↑
	one	two	three	four	five	six	seven	eight	nine[6]
	$(1)_{two}$	$(10)_{two}$	$(11)_{two}$	$(100)_{two}$	$(101)_{two}$	$(110)_{two}$	$(111)_{two}$	$(1000)_{two}$	$(1001)_{two}$

Thus by counting we see that $11 + 1001 = 1100$

<u>Method #2: The Monetary Model</u>

In terms of our \$1, \$2, \$4, \$8, etc. model, 11 means \$2 + \$1 and 1100 means \$8+\$4. Thus the question becomes "What denominations must we add to \$2 + \$1 to obtain \$8 + \$4. In this context:

-- we can begin by adding \$1 to \$2 +\$1. In that case we would now have

[6]Note that the words "one", "two", etc. are our own. The person in base-two would write them as 1, 10, 11, 100, etc.

\$2 + 2(\$1). This, in turn is the same as \$2 + \$2 or \$4.

-- and if we now add \$8 more we obtain \$8 + \$4.

-- therefore, we added \$1 + \$8 to \$2 + \$1 to obtain \$8 + \$4 as the sum.

Method #3: The "Traditional" Algorithm.

In this method we write the usual vertical form:

$$\begin{array}{ccccc} & 1 & 1 & 0 & 0 \\ - & & - & 1 & 1 \\ \hline \end{array}$$

To set the problem up properly, knowing that we can't subtract 1 from 0, we go to the 100's place and "borrow" the 1, thus converting the 0 in the 10's place into 10. That is:

$$\begin{array}{ccccc} & & 0 & & \\ & 1 & \not{1} & {}^{1}0 & 0 \\ - & & - & 1 & 1 \\ \hline \end{array}$$

We then borrow 1 from the 10's place, thus leaving us with 1 there and converting the 0 in the 1's place to 10. That is:

$$\begin{array}{ccccc} & & 0 & 1 & \\ & 1 & 1 & {}^{1}\not{0} & {}^{1}0 \\ - & & - & 1 & 1 \\ \hline \end{array}$$

Recalling that in this system 10 − 1 = 1, we now subtract in the usual way to obtain

$$\begin{array}{ccccc} & & 0 & 1 & \\ & 1 & \not{1} & {}^{1}\not{0} & {}^{1}0 \\ - & & - & 1 & 1 \\ \hline & 1 & 0 & 0 & 1 \end{array}$$

As a check, we see that $(1100)_{two} = 12$, $(11)_{two} = 3$ and $(1001)_{two} = 9$. Thus $(1100)_{two} - (11)_{two} = (1001)_{two}$ means the same thing as $12 - 3 = 9$, which is a true statement.

Multiplication and division are particularly simple in base-two arithmetic. The ease is based on the following.

<u>Practice Problem #4:</u>

What is the value of b if $(2 \times 10)_b = (20)_b$?

***Answer*: $b > 2$**

Solution:

The fact that the digit 2 appears indicates that the base has to be at least three (why?). Once we know that b has to be at least three, we know that in the multiplication tables $2 \times 1 = 2$. Hence, based on our adjective/noun theme, $2 \times b = 2b$[7]. No matter what the actual base is, in that base b will always appear as 10. In other words, in any base greater than 2, $2 \times 10 = 20$.

***<u>Notes</u>*:**

• One reason that many students like base-ten arithmetic is because it's so easy to think in terms of 10. While the importance of the number ten is basically limited to base ten arithmetic, the point is that in any number base, we would find it easy to think in terms of 10.

• In particular, this problem indicates that the nice property of "annexing" a 0 to multiply by 10 works in any number base.

• The result extends to all powers of 10. That is, $2 \times 100 = 200$, $2 \times 1000 = 2000$ etc.

• The fact that in base-two the only single digits are 0 and 1 means that the concept of repeated addition is particularly user friendly. For example:

[7]That is $2 \times b$ means the same thing as 2b's.

<u>Practice Problem #5</u>:

Again using the above tables, compute the product

$$(1011)_{two} \times (1001)_{two}$$

Answer: $(1011)_{two} \times (1001)_{two} = (1100011)_{two}$

Solution:

<u>Method 1: The Traditional Algorithm</u>

We use the addition and multiplication tables. They are simple to memorize. Namely

$1 \times 1 = 1$, $1 \times 0 = 0 \times 1 = 0 \times 0 = 0$; $1 + 0 = 0 + 1 = 1$, $1 + 1 = 10$.

$$
\begin{array}{cccccccc}
 & & & & 1 & 0 & 1 & 1 \\
 & & & \times & 1 & 0 & 0 & 1 \\
\hline
 & & & 1 & 1 & 0 & 1 & 1 \\
+ & & 1 & 0 & 0 & 0 & 0 & \\
 & & 0 & 0 & 0 & 0 & & \\
 & 1 & 0 & 1 & 1 & & & \\
\hline
 & 1 & 1 & 0 & 0 & 0 & 1 & 1
\end{array}
$$

<u>Method 2: The Distributive Property</u>

$$
\begin{array}{ll}
1011 \times 1001 & = \\
1011 \times (1000 + 1) & = \\
(1011 \times 1000)) + (1011 \times 1) & = \\
1011000 + 1011 \text{ (why?)} &
\end{array}
$$

Notice that the last sum is precisely what we obtained when we used the traditional algorithm. The previous result is reproduced below with the “0 rows” omitted and 0's added to the bottom row. That is:

$$
\begin{array}{ccccccc}
 & & & 1 & 0 & 1 & 1 \\
 & & \times & 1 & 0 & 0 & 1 \\
\hline
 & & & 1 & 0 & 1 & 11 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 1 & 1
\end{array}
$$

Once you feel comfortable with the concept of place value being independent of base ten, you can still use base ten to check your results. For example:

$(1011)_{two} = 1(8) + 0(4) + 1(2) + 1(1) = 8 + 2 + 1 = 11$

$(1001)_{two} = 1(8) + 0(4) + 0(2) + 1(1) = 8 + 1 = 9$

$(1100011)_{two} =$

$1(64) + 1(32) + 0(16) + 0(80 + 0(4) + 1(2) + 1(1) = 99$

Hence, translated from base-two to base-ten we see that we get the true statement $11 \times 9 = 99$

Practice Problem #6:

Again using the above tables, compute $(100011)_{two} \div (101)_{two}$**. That is, find the value of ? if** $(101)_{two} \times (?)_{two} = (100011)_{two}$**.**

Answer: $(100011)_{two} \div (101)_{two} = ((111)_{two}$

Solution:

An efficient way to begin might be to use rapid repeated addition. That is,

$$\begin{aligned} 101 \times 1 &= 101 \\ 101 \times 10 &= 1010 \\ 101 \times 100 &= 10100 \\ 101 \times 1000 &= 101000 \end{aligned}$$

101000 is greater than 100011. Hence we cannot subtract 1000 101's[8] (that is, 1000×101**). However we can subtract 100 101's. Namely:**

$$\begin{array}{rrrrrrl} 1 & 0 & 0 & 0 & 1 & 1 & \\ - & 1 & 0 & 1 & 0 & 0 & ((\ 100\ \ 101s) \\ \hline & & 1 & 1 & 1 & 1 & \end{array}$$

10 101's is equal to 1010. Hence we can subtract 1010 from 1111 to obtain:

$$\begin{array}{rrrrrrl} 1 & 0 & 0 & 0 & 1 & 1 & \\ - & 1 & 0 & 1 & 0 & 0 & (100\ 101\text{'s}) \\ \hline & & 1 & 1 & 1 & 1 & \\ & - & 1 & 0 & 1 & 0 & (10\ 101\text{'s} \\ \hline & & & 1 & 0 & 1 & \end{array}$$

[8]to avoid confusion you might be tempted to write a thousand 101's rather than 1000 101's. However, be careful. In the present context 1000 represents eight, not one thousand.

And since $101 \times 1 = 101$, we can conclude our repeated subtraction by writing

1	0	0	0	1	1	
–	1	0	1	0	0	(100 101's)
		1	1	1	1	
	–	1	0	1	0	(10 101's
			1	0	1	
		–	1	0	1	(1 101)
					0	

As a check, note that

$(101)_{two} = 5$, $(111)_{two} = 7$ and $(100011)_{two} = 35$.

Therefore

$(100011)_{two} \div (101)_{two} = (111)_{two}$

translates into

$35 \div 5 = 7$;

which is a true statement.

<u>Notes</u>:

• There are many nice things about base two arithmetic. One of them concerns the division algorithm. More specifically, in base ten arithmetic there are ten possible remainders when we divide one whole number by another. However in base two arithmetic there are only two possible remainders, 0 or 1.

• This greatly simplifies the computation when we perform the division algorithm. Namely if the partial dividend is greater than the divisor, then the divisor "goes into" the dividend just one time. Thus, unlike in greater bases, there is no need to use trial and error.

• With respect to the present problem, let's see how the division algorithm would have worked. We keep in mind that in base two $10 - 1 = 1$, and we start by writing the problem in the for

$$101 \overline{)100011}$$

We look at the dividend and see that it's at the third 0 where the dividend (1000) is greater than the divisor (101). Hence, without hesitation, we may place a 1 above the third 0 and write:

$$\begin{array}{r} 1 \\ 101 \overline{)100011} \end{array}$$

Then, in the "usual" way we multiply 101 by 1 to obtain 101 and we then subtract 101 from 1000 (needing only to remember that 10 – 1 = 1) to obtain:

```
                 1
1 0 1 | 1  0  0  0  1  1
      –    1  0  1
           -------
              1  1
```

We then "bring down" the 1 and proceed as before:

```
                 1  1
1 0 1 | 1  0  0  0  1  1
      –    1  0  1
           -------
              1  1  1
           –  1  0  1
              -------
                 1  0
```

Finally, we bring down the last digit in the dividend and repeat the above process

```
                 1  1  1
1 0 1 | 1  0  0  0  1  1
      –    1  0  1
           -------
              1  1  1
           –  1  0  1
              ----------
                 1  0  1
              –  1  0  1
                 -------
                       0
```

While we do not advocate using this method in class (other than for a form of enrichment), hopefully this demonstration helps clarify the long division algorithm by eliminates the need to use trial-and-error techniques.

3. The Method of Duplation:

Although place value wasn't to be invented until many centuries later, the ancient Egyptians, in effect, used base-two arithmetic in their algorithm for multiplying whole numbers. The algorithm was known as *the method of duplation.* It consisted of knowing only how to double a number and add. For example, to compute 37 × 19. They would start with the fact that 37 × 1 = 37; and they would then keep doubling 37 as shown in the following chart:

37 $\times$ 1	=	37
37 $\times$ 2	=	74
37 $\times$ 4	=	148
37 $\times$ 8	=	296
37 $\times$ 16	=	592

Knowing that 19 = 16 + 2 + 1, they would then check the rows in which 16, 2 and 1 appeared to obtain:

✓	37 $\times$ 1	=	37
✓	37 $\times$ 2	=	74
	37 $\times$ 4	=	148
	37 $\times$ 8	=	296
✓	37 $\times$ 16	=	592

They would then add 37, 74 and 592 to obtain 703. In summary:

$$37 \times 19 = 37(16 + 2 + 1) = (37 \times 16) + (37 \times 2) + (37 \times 1)$$

<u>A Note For the Classroom.</u>

Certainly the method of duplation is more tedious than the traditional algorithm for multiplication. However, it is an interesting way to emphasize how logic can convert complex problems into a series of simpler problems. Moreover, it is quite likely that many students will enjoy seeing this method and, in fact, might be tempted to check the results by doing the problems the "regular way".

As an additional aside, the method of duplation was modified under the Russian Tsars and became known as the Russian Peasant Method. The name came about because the Tsar wanted to find a simple method whereby one could perform multiplication knowing only how to multiply and divide numbers by 2 and add. The method began with the assumption that if you doubled one factor in a multiplication problem and halved the other, the product would not be altered. This method would work very simply if at least one of the factors was a power of 2. For example, suppose we wanted to compute 32 $\times$ 16. Using a horizontal array rather than a vertical array, we could write:

16	8	4	2	1
$\times$ 32	64	128	256	512

Notice that we got from one column to the next by halving one number and doubling the other. Hence in each column the product is the same as 32 $\times$ 16. In particular, we see at once that the product in the column

furthest to the right is 512. Therefore the product in each column, including the column furthest to the left (i.e., 16 $\times$ 32) must also be 512.

The major problem occurs when neither of the two numbers is a power of 2. In particular, let's revisit 37 $\times$ 19.

Step 1:

We proceed in the same way that we proceeded above but whenever there is a remainder when we divide by 2, we simply discard it! We would then obtain:

	19	9	4	2	1
×	37	74	148	296	592

Step 2:

We then place a check mark check next to each odd number in the top row:

	✓19	✓9	4	2	✓1
×	37	74	148	296	592

Step 3:

We then add the numbers that appear under the check marks to obtain the product of 37 and 19.

	✓19	✓9	4	2	✓1
×	37	74	148	296	592
	37	74			592

37 + 74 + 592 = 703

4. Revising History: Base Five Arithmetic:

In this section we will look at a bit of "revisionist history". Namely, we'll assume that because it was easier to keep track of numbers by counting on the fingers on one hand rather than on both hands, ancient civilizations decided to use five, rather than either ten or twenty, as the base of their counting systems. Thus, for example, the Romans would have used the letter X[9] to stand for five rather than for ten. In that way, XI would now represent $\underbrace{|||||}_{X}\ |$, or six.

Practice Problem #7

In terms of the revised Roman numeral system, what number (written in our traditional base ten format) is represented by XXXI?

Answer: **sixteen**

Solution:

In this system, X is an abbreviation for I I I I I. Hence XXXI is an abbreviation for I I I I I I I I I I I I I I I I. Since the number that is represented by the Roman numerals does not depend on how the numerals are grouped, we move the second group of five tally marks closer to the first group of five and we can move the single tally mark at the end closer to the third group of five; thus giving us the arrangement: I I I I I I I I I I I I I I I I, which we recognize as naming the number we write as 16.[10]

Notes:

• If we simply wanted to obtain the answer almost by rote, we would note that since X stood for five, XXX would stand for 3 five's or fifteen, and $15 + 1 = 16$. However such an approach defeats the purpose of what a number base really means; at least in the sense that once we say that X stands for five, there should be no need to refer to base ten to do arithmetic. In other words, writing $3 \times 5 = 15$ and/or $15 + 1 = 16$ make it seem that we need base ten arithmetic in order to do arithmetic in base five.

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[9]We changed the color of X in order to help avoid confusion. That is, it would be confusing to have X stand for both five and ten. However it is easy to distinguish between X and X. We could have chosen a completely different symbol to represent five but we want to maintain the flavor of what happened historically.

[10]Notice the difference between saying "sixteen" and writing 16. The word "sixteen" means the number which when represented by tally marks looks like | | | | | | | | | | | | | | | |. The tally marks may be regrouped in many ways and when we write 16 it assumes that we have grouped them into one batch of ten and another batch of six.

• The point is that if place value had been developed as a continuation of our "revisionist history", there would have been no numerals 5, 6, 7, 8 or 9 and only the digits 0, 1, 2, 3 and 4 would have been used. In particular, 10 now would represent five and 100 would represent twenty five (that is, five five's)[11]. To see this in a rather interesting way, imagine an automobile odometer that used base five. Each gear would have five faces named 0, 1, 2, 3 and 4. Looking at the gear in the ones place, it would keep repeating the cycle 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 0....Thus on a brand new car, after it had been driven for 4 miles the odometer would read 000004; and after the next mile it would read 000010.

• The idea of an odometer also applies to bases that are greater than ten. For example if we were to construct a base-twelve odometer, each gear would have to have twelve faces, all named by a single digit because the first multi-digit numeral, 10, now stands for 1 twelve and no ones; that is, $(10)_{twelve} = (12)_{ten}$.

• In the base-twelve system, the numbers, ten and eleven would have to be represented by single digits, such as T (for "ten") and E (for "eleven"). One would then have counted: 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1T, 1E, 20, ...[12]

• As a classroom activity, using the odometer as a model is probably a good way to visualize how counting would proceed in any given number base.

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[11]When we write five (as opposed to writing 5) we are talking about a number that is independent of any base. In other words, in terms of tally marks, 5 represents | | | | |. In terms of our own place value system this number is represented by the numeral 5 while in our revised Roman numeral system it is represented by the numeral 10. To avoid confusion when we use a base other than ten we should read 10 as "one zero" in order not to confuse it with the number ten.

[12]Note that those of us who live in the base-ten numeral system no more need to use T and E than the person who lives in base five would need to use the numerals, 5, 6, 7, 8 and 9.

Practice Problem #8:

Suppose the Romans had decided to let X represent | | |. What number (written as a base-ten numeral) would be represented by XXI?

Answer: **7**

Solution:

If X represents | | |, XX represents | | | | | |.

Hence, XXI represents | | | | | | |. Counting the tally marks we see that there are seven of them; and in base ten, seven is represented by the numeral 7.

Note

The notation $(\ )_b$ is used to indicate that the numeral inside the parentheses is representing how the number would be written in base b.

Practice Problem #9:

Write the number represented by $(15)_{ten}$ in the form of a base-seven numeral.

Answer: **$(21)_{seven}$**

Solution:

In terms of tally marks, $(15)_{ten}$ means:

| | | | | | | | | | | | | | |

However to indicate that we're "trading in" by seven's we could move three of the tally marks from the group of ten and annex them to the group of five to obtain, as shown below, one group of seven tally marks and a second group with eight tally marks:

| | | | | | | | | | | | | | |

The second group of tally marks can then be rewritten as one group of seven and another group with just one. That is:

| | | | | | | | | | | | | | |

In base-seven, X would represent | | | | | | |; therefore | | | | | | | | | | | | | | | would have been written as XXI and in place value notation we would write this as $(21)_{seven}$.

Notes:

• This might be a nice place to have students visualize the result in terms of a base-seven odometer. In this odometer, there are only the seven faces 0, 1, 2, 3, 4, 5 and 6. After 6 the numeral 0 comes up and causes the numeral on the gear to its left to increase by 1. So, for example:

Miles Driven	Base-Seven Reading	Base-Ten Reading
none	0000	0000
one	0001	0001
two	0002	0002
three	0003	0003
four	0004	0004
five	0005	0005
six	0006	0006
seven	0010	0007
eight	0011	0008
nine	0012	0009
ten	0013	0010
eleven	0014	0011
twelve	0015	0012
thirteen	0016	0013
fourteen	0020	0014
fifteen	0021	0015

• Students may find it easier to think in terms of money. With respect to base seven imagine a monetary system in which the only denominations are $1-bills, $7-bills, $49-bills etc. (where seven of any denomination can be exchanged for one of the next greater denomination). In this system we can buy a $15 item by using two $7-bills and one $1-bill. That is, $(15)_{\text{ten}} = (21)_{\text{seven}}$

5. A Note on Base One Thousand

The idea of counting by thousands, millions, billions, trillions etc. is, in effect, using one thousand as a number base. This may seem a bit far-fetched but to see what we mean recall our previous discussion when we talked about viewing 5 tens and 13 ones as 513. We said that in order not to confuse this with 5 hundreds, 1 ten and 3 ones, we could represent 5 tens and 13 ones by using parentheses and writing 5(13). That is, everything within parentheses is treated as one number. In that context, for example, (51)(213)(4) would stand for 51 hundred, 213 tens and 4 ones.

What may be somewhat of a surprise, notation such as 513,723,912,415 is a somewhat disguised version of base-one thousand arithmetic. Namely, suppose we used parentheses rather than commas and wrote the number as (513)(723)(912)(415), which would be read as "513 billion, 723 million, 912 thousand and 415 ones. In this context, we have the thousand numerals 000 through 999 as our adjectives and the denominations are units, thousands, millions and billions. Thus instead of trading in ten of one denomination for one of the next higher denomination we trade in a thousand of one denomination for one of the next higher denomination.

Whereas it was relatively simple in base-ten to memorize the tables through 9 and then let place value take over, it would be a monstrous task to use a place value system in which we had to memorize the tables through 999. That's why we preferred to use commas and still work as if we were in base-ten. From the point of view of using an odometer, we would have to have each gear have 999 faces and the next number after 999 on a gear would then be 000. So, for example, after 2(999)(999) would come 3(000)(000). etc.

6. "Numbers Versus Numerals" Revisited

Consider the "pseudo-syllogism":

> ***Cat is a three-letter word.***
> ***All cats chase mice.***
> ***Therefore, some three-letter words chase mice*.**

Our "silly" conclusion follows from the fact that in actuality the first two statements have different subjects. Namely, in the first statement we are referring to the *word*, "cat"; while in the second statement we are referring to the *animal*, cat. In other words, it would have been more precise to have written:

> ***"Cat" is a three-letter word.***
> ***All cats chase mice.***

As we have seen throughout our course, there is a big difference between a number (which, unless it is clear from context, we usually spell) and the *numeral* that names that number. For example we would write "eight" when we talked about the number but we would write 8 (or 5 + 3, etc.) when we were talking about the numeral that represented the number eight.

Thus when we study number theory we have to distinguish it from the study of "numeral" theory. To emphasize this point in more detail., let's look at the following examples:

<u>Practice Problem #10:</u>

Is the following explanation correct?

"Because $(32)_{five}$ ends in an even digit, it must represent an even number".

***Answer:* No**

Solution:

$(32)_{five}$ means $3(\text{five's}) + 2(\text{ones}) = (15)_{ten} + (2)_{ten} = (17)_{ten;}$ and $(17)_{ten}$ is an odd number.

<u>Notes:</u>

- **In terms of tally marks we may view $(32)_{five}$ as | | | | | | | | | | | | | | | | |. If we now group the tally marks in pairs, we obtain | | | | | | | | | | | | | | | | |.**

The point is that even and odd are defined in terms of divisibility by two.

•When we group seventeen tally marks in sets of two, there is one tally mark that is left unpaired. The seventeen tally marks do not know how we intend to group them; but no matter what base we use (or even if we don't think in terms of a number base) they always represent an odd number.

• This result holds for any odd base. For example:

-- $(32)_{seven} = (23)_{ten}$

-- $(32)_{nine} = (29)_{ten}$

-- $(32)_{eleven} = (35)_{ten}$

In terms of the above problem we have shown that some numeral properties depend on the base we are using. However, there are times when a property belongs to a number, independently of what base the number is being viewed. For example:

<u>Practice Problem #11</u>

In what number bases is nine a perfect square?

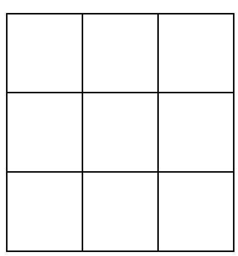
Answer: **Every number base**

Solution

Let's look at the number, nine. No matter what base we choose, nine will always be a perfect square. Namely, by the judicious use of tally marks (the number of which is independent of place value), we see that nine tally marks can be arranged to make a three by three square. That is:

<u>Note:</u>

In the earlier grades students prefer squares to tally marks. Thus you might want to replace the diagram above by



Of course, the way this result would be expressed in place value depends on the base we are using. For example,

-- In base five, nine would be written 14 and we would then say that $3^2 = 14$.

-- In base three, three would be written as 10 and nine would be written as 100. Hence in base three the fact that nine is a perfect square would be written as $10^2 = 100$.

At any rate this ends our discussion of different number bases, at least for the time being.

In the next lesson we shall begin our discussion on fractions.