CALCULUS IN EVERYDAY LIFE

PART 2 DIFFERENTIAL CALCULUS

OR

WHY UNDERSTANDING RATES IS SO IMPORTANT

(AN ARITHMETIC APPROACH)

1. What Is Differential Calculus ?

The word "calculus" is a Latin word that means "stone". The Western World's forerunner of the abacus was a system of vertical lines that denoted the various place value denominations; and stones were used as markers to indicate the number of each denomination. Thus, the verb "to calculate" originally meant to do arithmetic by using stones. In this sense, then, all of computational mathematics could have been called calculus.

So perhaps the best place to begin our discussion is with the question:

> **What do we "calculate" in differential calculus that we do not "calculate" in other computational mathematics courses?**

As a start we will say that differential calculus may be viewed as the study of *instantaneous rate of change*. However because the topic of rates has already been discussed in the elementary school mathematics curriculum (usually when fractions are taught), the study of differential calculus begins when we try to define what "instantaneous" means.

- **By way of review, a rate of change (or, more simply, a rate) is usually expressed in the form of two nouns separated by the word "per". We deal with rates when we want to answer the question "how fast?" rather than "how much?"**

- **For example, speed is a rate. It is the rate of change of the distance traveled with respect to the time it took to travel this distance; and is expressed in such units as "miles per hour", "feet per second", "meters per minute", etc.**

- **The point is that by themselves neither the distance nor the time it took to travel the distance tell us how fast an object moved. Rather we have to know the ratio (i.e., rate of change) of the distance traveled with respect to the time it took. For example if an object travels at a constant speed and goes 100 miles in 4 hours, it is moving only at a rate of 25 miles per hour. On the other hand if an object traveling at a**

constant speed goes only 1 mile but it takes only 30 seconds, it is moving at a rate of 120 miles per hour!

• In their simplest form, rates are *constant*. A constant rate, as the name implies, is a rate that never changes. For example, there are always 12 inches per foot, 4 quarts per gallon, 100 centimeters per meter, etc. Word problems that we deal with in elementary mathematics courses often deal with constant rates. Thus a typical problem might read:

"If apples cost $1.30 per pound, how much will 5 pounds of apples cost?"

In this question it is assumed that every time we buy one more pound of apples, we have to spend another $1.30. In other words, the rate at which we are spending money remains constant; namely, $1.30 per pound. Thus to find the cost of 5 pounds of apples, we need only multiply $1.30 by 5.

• However, not all rates are constant. To get an idea as to what this might mean, critique the following logic.

> **"A man drives 200 miles in 4 hours. Therefore it took him 2 hours to drive the first 100 miles."**

Hopefully the mistake in the above logic is obvious. More specifically, how likely is it that the automobile traveled for 4 hours without ever changing its speed? For example, because of traffic situations, perhaps the man drove 120 miles during the first 2 hours but only 80 miles during the second two hours.[1]

• In cases where the rate is not constant, we often talk about the average rate.

Definition:

Suppose a quantity A is related to another quantity B, By the average rate of change of A with respect to B, we mean the total change in A divided by the corresponding change in B.

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[1]The reason that constant rate problems occur in elementary mathematics textbooks is that they are easy to handle. That is, when rates are not constant we can't use such simple formulas as

"Distance $=$ Rate $\times$ Time".

because if the rate is *not* constant, what number shall we use as "*the* rate" ?

For example, based on this definition, we can say for sure that if a man drove 200 miles in 4 hours, his average speed for the trip was 50 miles per hour. In this case, we divide the distance the man drove (200 miles) by the time it took (4 hours) to conclude that the average speed was 50 miles per hour.[2]

The interesting point is that no matter how it happens, if an object travels 200 miles in 4 hours, its *average* speed is 50 miles per hour.

For example, it might have traveled 100 miles during the first hour and then took the remaining 3 hours to go the other 100 miles. Or it might have traveled the first 150 miles in 50 minutes; then stopped for 3 hours; and then went the remaining 50 miles in the final 10 minutes. Or, as unlikely it might be, it might have traveled at a constant rate of 50 miles per hour for the entire 200 miles trip.

• Perhaps an example that might be more relevant to students is what is meant by "the average test score". Suppose, for example, that a student, after completing five equally weighted tests, has an average score of 80 (points per test). In that case all we know for sure is that the student scored a total of 400 points on the 5 tests. That is, the average test score is obtained by taking the total number of points and dividing it by the total number of tests. Thus:

$$80 = (\text{total number of points}) \div 5;$$

or

$$\text{total number of points} = 80 \times 5 = 400$$

So while it's possible (but probably unlikely) that the student received a score of 80 on each of the 5 tests, any combination of test scores that totaled 400 points would have yielded the same average score.

Getting back to our main point:

Once we assume that students understand what is meant by rate of change and average rate of change, all that is needed in order to introduce calculus is define what is meant by "instantaneous". In that context, elementary school mathematics plays a very important role in the study of mathematics in general and differential calculus in particular.

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[2]If the word "average" was omitted, we would probably have had to assume that the object traveled at a constant speed (otherwise the problem would have been too difficult to solve by the techniques of elementary mathematics).

2. The Concept of an Instant

It is when we try to come to grips with what we mean by an *instant* that the study of calculus begins. To appreciate the difficulty of defining an instant, try to answer the following question:

Where was the man at the instant he jumped off the bridge?

If the man is already in the air then he must have jumped previously (for example, you can't be 2 inches off the ground before you are 1 inch off the ground); and if he is still on the bridge, then he hasn't jumped off yet!

In other words, there is a unique point in time (that is, an instant) where just before that the man is still on the bridge and just after that he is already in the air. [3]

How can we locate when this "instant" occurred? One way might be to use the "roll of movie film" notion that we discussed in Part 1. That is, we make a movie of the event with a timer that tells us when each frame was made.

-- Suppose that in the frame which reads 3 seconds the man is still on the bridge but that in the frame that reads 4 seconds he is already in the air. Then we know that the time at which he jumped off was later than 3 seconds after the tape began but earlier than 4 seconds after the tape began.

-- We could then look through the frames that are in the time interval between 3 and 4 seconds to see where he was at other times; for example, say, at the end of 3.5 seconds. If he has already jumped, then we know that he jumped after 3 seconds but before 3.5 seconds. We can repeat this procedure as often as we wish and get closer and closer to the exact time at which he jumped.

However, sooner or later, we get to a stage where our camera cannot measure the time any more closely. That is, there is a time lapse, no matter how brief, between any two consecutive frames. As a result we would not be able to measure any times that were less than this time lapse.

For example, if the time lapse was 0.01 seconds, how would we find consecutive frames that were timed at 3 seconds and 3.0001 seconds? And if we could improve the technology so that the time lapse was only 0.00001 seconds, how would we

[3] In a more personal sense one can say that an instant is that unique moment in time when it is the oldest you have ever been but the youngest you will ever be again,

find the frames that were timed at 3 seconds and 3.0000001 seconds?

Do you see the problem with this technique? Namely, we assume that time is *continuous*; that is, that there are no "gaps" in time. Yet a camera cannot record the time continuously.

In other words, between one frame and the next, some time (no matter how small) has elapsed and we eventually get to the point where the available technology is not able to construct a video timer that can detect the exact time.

Note:

The above discussion illustrates the difference between what mathematicians call *continuous* and *discrete variables.* While time is a continuous variable, the number of frames isn't. For example, while there are *infinitely* many time intervals that are longer than, say, 1 second but shorter than 2 seconds; there are no frames between the 1st and the 2nd. In other words, increasing values of a discrete variable can be listed consecutively (that is, in a way whereby there are no other values between any two listed values). On the other hand, for a continuous variable, no matter how close in value two different numbers are, we can always find another number that lies between them.

This brings us to an important distinction between what we might call the "real" world and the "ideal" world. For example, in the "real" world if we know that the man was still on the bridge at the end of 3 seconds but that he was in the air at the end of 3.1 seconds, we would be inclined to say that we knew when he jumped. In essence, if anyone were to say that we still didn't know the exact time, we would be tempted to say that our answer was "close enough". Yet while the "close enough" answer suffices in most practical cases, the fact is that there is value in trying to find the exact answer. That is, there are times when "close enough" isn't close enough! [4]

[4] For example, suppose you want to buy one item that is priced at 3 for $1. Rounded off to the nearest cent, you should only have to pay 33¢ for the one item. However, the store will charge you 34¢ (or more!) because at 33¢ each the price would only be 3 for $0.99. Yet how likely is it that the difference between 33¢ and 34¢ will deter you from buying that one item? However, if you were a buyer for a large chain of stores and you were buying 1,000,000 items the difference between 33¢ per item and 34¢ per item would amount to $10,000!

3. An Important Note on Infinite Versus "Very Many":

The fact that there is always a time lapse between two consecutive frames of movie film brings us to what we might call "uncharted waters". Namely, we would need to have an infinite number of frames if we were to have any chance of finding the exact time the man jumped. The problem is that all of our previous experience has dealt with finite numbers, some of which were very big.

Here is an interesting way to see the difference between "very, very many" and "infinitely many".

> **Suppose there are 5 people in a room and they each change their name. Does that change the number of people who are in the room? And the answer is "Obviously not!". And if there were 1,000,000,000 people in a room (it would have to be a big room) and they each changed their names, it still wouldn't change the number of people who were in the room.**
>
> **Now suppose instead we look at the set of whole numbers (i.e., 0, 1, 2 3, 4,...). and we multiply each of the numbers by 2, thus changing the name of the number to twice its present value. Amazing as it may seem, we now have converted the set of whole numbers into the set of even whole numbers, without consciously removing any whole number from the set! More specifically:**
>
> **Original Set: 0 1 2 3 4 5 6 7 8 9 10 11 12 ...**
>
> **↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓**
>
> **The New Set: 0 2 4 6 8 10 12 14 16 18 20 22 24 ...**
>
> **In terms of our adjective/noun theme, we can make the above observation seem even more amazing. namely suppose we ask each number to choose a noun to modify and the noun they choose is "billion". Then 1's new name is "1 billion"; 2's new name is "2 billion" etc. In this way what used to be the set of all whole number is now the set that consists only of the multiples of a billion. Where have all the other numbers gone???**

This is not a paradox. Rather it simply demonstrates that a property that is obeyed by a finite set of members (no matter how large) need not be obeyed by an infinite set of numbers. And, perhaps of even greater importance, we have to be prepared to come to grips with ideas that may seem to be counter-intuitive.

4. What Is Instantaneous Speed?

The concept of an "instant" is further complicated when we introduce the notion of instantaneous speed. For example, suppose instead of asking the question, "Where was the man at the instant he jumped off the bridge?", we had asked "How fast was the man moving at the instant he jumped off the bridge?"

The problem is that speed is defined to be the rate of change of distance with respect to the corresponding change in time. However if no time transpires, there is no change in distance; and hence the definition of instantaneous speed becomes the study of "0 divided by 0".

To see this in terms of our "roll of movie film" model, suppose we actually found the frame that represents the instant the man jumped off the bridge.

-- A frame is inanimate. That is, no motion takes place during a frame.

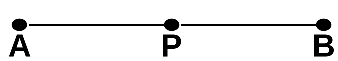
-- The film is a collection of frames.

-- Hence no motion takes place in the film.

In essence it means that to know how fast a person was moving at a particular instant we have to know something about what happened "just before" and "just after" the given instant.[5]

In more precise terms we have to know what is happening in a time interval that includes the given instant.

With this in mind we are able to compute the average speed for intervals that include the specific instant in question. For example suppose an object is moving along the x-axis in the positive direction and we want to know what its speed is at the instant the object is at the point P. We can place an observer (A) to the left of P and another observer (B) to the right of P. Pictorially:



We can measure the distance between A and B and the time it took for the object to move from A to B; and in that way we can compute the average speed of the object during the time interval from A to B. However our goal was to find the instantaneous speed and there is no reason to believe that the average speed in this case is a good approximation to the speed of the object when it is at the point P.

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[5] This may be easier to visualize in terms of geometry. Given any point P in the plane,,, there are many curves that pass through it. To determine the shape of a particular curve that passes through P we have to know what the curve looked like just before it got to P and just after it passed through P .

However as the distance between A and B diminishes, we expect that the average speed of the object in going from A to B becomes a better and better approximation to the speed of the object when it is at the point P.

A P B

Thus our intuition would seem to tell us that the approximation would become the exact speed when the distance between A and B decreased to 0.

P
A
B

However as we discussed previously if the distance between A and B is 0, we are back to the "0 divided by 0" situation.

Hence we want to compute the average speed of the object between A and B as the distance between A and B gets arbitrarily close to 0, without ever becoming 0.

However if there is a non-zero distance between A and B it means they can move even closer without coinciding with one another. In other words, we would need infinitely many pairs of observers to measure the average speed of the object as the distance between A and B kept getting less.

Historical Note:

The question of instantaneous speed was first raised many centuries ago. Zeno of Elea was a Greek Philosopher who lived in the 5^{th} century B.C. and one of the things for which he is famous is his paradox [6] concerning the claim that an arrow in flight never moves. Zeno's argument was that since it takes time to move, in a given instant the arrow is not moving (because no time transpires in an instant);and since time is a collection of instants, the arrow never moves.[7] The resolution of Zeno's paradox resulted in the observation that an infinite number of events could transpire in a finite amount of time.

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[6] A paradox is something that isn't true but seems to be logically correct.

[7] It is called a paradox because Zeno, as did everyone else, knew that the arrow moved. yet his logic seemed to prove that it couldn't be moving.

5. Calculating Instantaneous Speed

Very often in mathematics when we sense that a concept exists but don't know how to define it explicitly, we look at the properties the concept is to have and then see what this implies logically.

For example, with respect to Zeno's paradox, we know that an arrow in flight has a speed at any instant that it is flight, but we don't know how to compute it explicitly. So let's make some assumptions that we believe are true. In particular:

- **An object has a speed at each instant that it is in motion.**
- **The average speed of an object during a particular time interval is, by definition, the quotient we obtain when we divide the distance the object traveled during this time interval by the time it took the object to travel this distance.**
- **Unless an object is traveling at a constant speed[8], the average speed of an object during a given time interval is greater than the least speed of the object during the time interval but less than the greatest speed of the object. [9]**
- **If the object is accelerating during the entire time period (that is, it is moving faster and faster), the least speed occurs at the beginning of the time interval and the greatest speed occurs at the end of the time interval.**

To help get around the question of "close enough" we introduce the notion of a function (formula) to replace the need for having infinitely many observers or infinitely many frames of a motion picture reel.

Notice that, unlike the movie frames, the formula allows us to compute the position of the object at every instant of time. In particular we will be able to find the average speed of the object during any time period we desire.

With the above points in mind and armed with a hand held calculator we are now ready to tackle a problem such as the one below.

[8]If the object is traveling at a constant speed, its average speed, its instantaneous speed, its least speed and its greatest speed are all the same.

[9]This might be easier to visualize in terms of test scores. Suppose that you took 5 tests and had an average of 80 points per test. That means that you scored 400 points. If all of your scores had been less than 80 points, you would have scored less than 400 points; and if all of your scores had been greater than 80 points you would have scored more than 400 points.

Illustrative Problem #1:

An object is moving along the x axis in such a way that after t seconds it is x feet from the origin where

$$x = t^3, \text{ where } 0 \le t \le 4.$$

What is the velocity [10] of the object at the instant $t = 2$?

Note:

We include a constraint such as $0 \le t \le 4$ because there is usually a limit to how long the rule that governs the motion applies. For example, let's suppose the rule applied for 1,000 seconds (which is a little less than 17 minutes). It would mean that during the 1,000th second of its motion, the object would have traveled 2,997,000 feet!

More specifically when $t = 999$, $x = 999^3 = 997{,}002{,}999$[11] (feet); and when $t = 1{,}000$, $x = 1{,}000^3 = 1{,}000{,}000$ (feet). This means that during the 1,000th second the object would have traveled:

$$1{,}000{,}000 \text{ feet} - 997{,}9999 \text{ feet} = 2{,}997{,}001 \text{ feet}.$$

Very few objects travel at such a speed! [12]

Keep in mind that so far we are only able to compute average speed. We know that the average speed is between the least speed and the greatest speed in any time interval during which the object is moving. The chart below indicates that the object is accelerating (that is, during each second it travels further than it did in the previous second).

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[10] If the speedometer on your car reads 40 mph, that is your speed regardless of the direciton n which you are going. Velocity is speed in a given direction. for example it makes a difference whwelther you are travelig 40 mph due north or 40 mph due east; even though in either case your speed would be 40 miles per hour. In this example speed and velicity are the same because we are told that the obect is moving in th edirection of th eppositive x-axis.

[11] This is why we suggest being armed with a calculator. Namely it would be quite laborious (but not impossible) to compute 999^3

[12] Of course there are notable exceptions. For example light travels at a speed of 186,000 milers per second; or in terms of feet, 256,608,000 feet per second.

t	x ($= t^3$)	distance traveled per second (average speed)
0	0	
1	1	1 foot (i.e., 1 foot $-$ 0 feet)
2	8	7 feet (i.e.,8 feet $-$ 1 foot)
3	27	19 feet (i.e.,27 feet $-$ 8 feet)
4	64	37 feet (i.e.,64 feet $-$ 27 feet)

Let's represent the velocity of object when the time is 2 seconds by $v_{t=2}$ Notice that $t = 2$ is the end of the time interval from $t = 1$ to $t = 2$; and since the object is accelerating it means that $v_{t=2}$ must be greater than the average speed during the 2nd second (i.e. from $t = 1$ to $t = 2$) . From the above chart we see that the average speed during the time interval from $t = 1$ to $t = 2$ is 7 feet per second; and therefore:

$$v_{t=2} > 7 \text{ feet per second.}$$

At the same time $t = 2$ is the beginning of the time interval from $t = 2$ to $t = 3$ and therefore the velocity of the object when $t = 2$ must be less than the average speed of object during the time interval between $t = 2$ and $t = 3$. From the above chart we know that the average speed during the time between $t = 2$ and $t = 3$, which is 19 feet per second. In other words::

$$v_{t=2} < 19 \text{ feet per second.}$$

Therefore what we can say for sure is that at the instant $t = 2$ the exact speed (velocity) of the object is between 7 feet per second and 19 feet per second. While there are many speeds that are between 7 feet per second and 19 feet per second, we have still managed to limit the possibilities.[13]

[13]Notice that while we do not what the exact speed is, we have logically deduced that no mattter what it is, it has to be btween 7 feet per seodnd and 19 feet per second. This is a statement of fact; not a statement that is "approximately" true.. it is not an approximation.

We can refine our approximation by choosing a shorter time interval. For example, if we let t_1 denote any time that is less than 2 seconds, we can generalize what we did above by saying that the exact speed of the object at the instant $t = 2$ must be greater than the average speed of the object during the interval between $t = t_1$ and $t = 2$.

In a similar way it t_2 is any time that is greater than 2 seconds, we can generalize what we did above by saying that the exact speed of the object at the instant $t = 2$ must be less than the average speed of the object during the interval between $t = 2$ and $t = t_2$.

So to "cut to the chase" let's pick a much shorter time interval by letting $t_1 = 1.999$ and $t_2 = 2.001$[14]

By the definition of average speed the average speed of the object between $t = 1.999$ and $t = 2$ is the distance traveled during this time ($2^3 - 1.999^3$) divided by the time interval between $t = 1.999$ and $t = 2$. More specifically:

-- When $t = 2$, $x = 2^3 = 8$ (feet)

-- When $t = 1.999$, $x = 1.99^3 = 7.988005999$ (feet)

-- Therefore the distance the object travels during the mine interval between $t = 1.999$ seconds and $t = 2$ seconds is

$$8 \text{ feet} - 7.988005999 \text{ feet} = 0.011994001 \text{ feet}$$

And the time it took to travel this distance is

$$2 \text{ seconds} - 1.999 \text{ seconds} = 0.001 \text{ seconds}$$

Hence the average speed during the time interval between $t = 1.999$ seconds and $t = 2$ seconds is:

$$\frac{0.011994001 \text{ feet}}{0.001 \text{ seconds}} = 11.994001 \text{ feet per second}$$

Therefore

$$v_{t=2} > 11.994001 \text{ feet per second.}$$

[14]To make it easier to keep track of things we have choose t_1 and t_2 to equidistant from $t = 2$. However there is no need to do so. For example,we could have let $t_1 = 1.987$ and $t_2 = 2.013$. All that matters is that t_1 is less than 2 and t_2 is greater than 2. Whaqt is important is that we look at an interval that ends at $t = 2$ and another interval that begins at $t = 2$.

In a similar way, by the definition of average speed the average speed of the object between $t = 2$ and $t = 2.001$ is the distance traveled during this time $(2001^3 - 1.999^3)$ feet, divided by the time interval between $t = 2$ and $t = 2{,}001$. More specifically:

-- When $t = 2$, $x = 2^3 = 8$ (feet)

-- When $t = 2.001$, $x = 2.001^3 = 8.012006001$ (feet)

-- Therefore the distance the object travels during the mine interval between $t = 2$ seconds and $t = 2001$ seconds is

$$8.012006001 \text{ feet} - 8 \text{ feet} = 0.012006001 \text{ feet}$$

And the time it took to travel this distance is

$$2.001 \text{ seconds} - 2 \text{ seconds} = 0.001 \text{ seconds}$$

Hence the average speed during the time interval between $t = 2$ seconds and $t = 2.001$ seconds is:

$$\frac{0.012006001 \text{ feet}}{0.001 \text{ seconds}} = 12.006001 \text{ feet per second}$$

Therefore

$$v_{t=2} < 12.006001 \text{ feet per second.}$$

In summary,

$$\boxed{11.994001 \text{ feet per second} < v_{t=2} < 12.006001 \text{ feet per second}}$$

Rounded off to the nearest tenth of a foot per second, both 11.994001 feet per second and 12.006001 feet per second become 12 feet per second. Therefore while we still don't know the exact speed when $t = 2$, we can say for sure that rounded off to the nearest tenth of a foot per second it is exactly 12 feet per second.

For many real-world applications the above estimate is “good enough”. However if we wanted to have even more accuracy we could continue to make the interval smaller and smaller. For example (and we leave the details for you to fill in) if we look at the time interval between $t = 1.999999$ and $t = 2$ and at the time interval between $t = 2$ and $t = 2.000001$, we see that:

$$\boxed{11.999994000001 \text{ feet per second} < v_{t=2} < 12.00000600001 \text{ feet per second}}$$

By this time, even though we have not proven it to be true, we are most likely to believe that at the instant $t = 2$, the speed of the object is 12 feet per second. However at the very least we can now be sure that to the nearest hundred thousandth of a foot per second, the exact speed of the object at the instant $t = 2$ is exactly 12 feet per second.

An Important Note:

Notice that approach we used here is slightly different from our qualitative description of having an observer on each side of the point in question. More specifically if we had followed our qualitative description, we would have had one observer stationed, say, at $t = 1.99$ and the other observer stationed at $t = 2.01$ and then computed the average speed of the object during the time interval from 1.99 seconds to 2.01 seconds.

Instead we used three observes; namely one at $t = 1.99$, one at $t = 2$ and one at $t = 2.01$. We then computed two average speeds; one between 1.99 seconds and 2 seconds, and the other between 2 seconds and 2.01 seconds. In most cases, the two methods are equivalent. However, there are "pathological" exceptions that make it safer to use three observers rather than 2 observers. For example, suppose you are shipping a package and the cost is \$3 for packages that weigh less than 1 pound but \$5 for any package that weights 2 pounds but less than 4 pounds.

Clearly using either method we see that the cost is constant for any weight that is less than 2 pounds or if the weight is at least 2 pounds but less than 4 pounds (i.e., \$5). However, the difference in the two methods occurs when the weight in questions "approximately" 2 pounds.

More specifically, the weight of a package that weighs slightly less than 2 pounds is \$3 while the weight of a package that weighs 2 pounds or slightly more than 2 pounds is \$5. Hence the change in cost over the weight interval from say 1.99 pounds to 2.01 pounds is \$2 (i.e., \$5 − \$3) and since the weight interval is 0.02 pounds the average cost of a package in that weight interval would be \$2 ÷ 0.02 pounds or \$100 per pound. On the other hand if the interval had been between 1.99999 pounds and 2.00001 pounds, the change in cost would still be \$2 but the size of the weight interval would be 0.00002 pounds; and hence the average cost per pound during this time interval would be an astounding \$2 ÷ 0.00002 pounds or \$100,000 per pound. This is what we mean by a "pathological" situation. Namely as the size of the interval around $t = 2$ decreases, the average cost per pound increases without bound.

On the other hand using our three observers' approach we see that the problem breaks down into two parts; one part consisting of what happens when the weight in question is less than 2 pounds and the other when the weight in question is greater than or equal to 2 pounds.

6. Generalizing Our Discussion.

We looked at the relationship $x = t^3$ simply as an illustration of the technique we may use for pinpointing the speed of an object at any instant. More generally the technique we used works for any relationship. in fact, if the function is represented by a key on our calculator, we don't even have to know the properties of the function in order to find the exact speed of an object. For example, suppose we had never studied exponential functions but your calculator had an x^y key.

We could compute the value of, say, $2^{3.4}$ by using the following sequence of key strokes:

$\boxed{2}\ \boxed{x^y}\ \boxed{3}\ \boxed{.}\ \boxed{4}\ \boxed{=}$

So suppose instead of $x = t^3$ the relationship had been $x = 3^t$ and we were given the following problem to solve.

Illustrative Problem #2:

An object is moving along the x-axis in such a way that after t seconds it is x feet from the origin where

$$x = 3^t, \text{ where } 0 \leq t \leq 4.$$

What is the velocity of the object at the instant $t = 2$?

Notice that the object is again accelerating as we can see from the chart below.

t	x ($= 3^t$)	distance traveled per second (average speed)
0	1[15]	
1	3	2 feet (i.e., 3 foot − 1 foot)
2	9	6 feet (i.e., 9 feet − 3 feet)
3	27	18 feet (i.e., 27 feet − 9 feet)
4	81	54 feet (i.e., 81 feet − 27 feet)

We also see from the chart that the average speed of the object during the time interval from $t = 1$ to $t = 2$ is 6 feet per second; and the average speed of the object during the time interval from $t = 2$ to $t = 3$ is 18 feet per second. Therefore, since the object is accelerating:

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[15]In our discussion of exponential functions we demonstrated why and (non-zero) number to the 0^{th} power is 1. However ever if we had never studied exponential functions the sequence of key strokes $\boxed{3}\ \boxed{x^y}\ \boxed{0}\ \boxed{=}$ would have yielded the result that $3^0 = 1$.

$$6 \text{ feet per second } < v_{t=2} < 18 \text{ feet per second}$$

We now have a rather rough estimate for the exact speed when $t = 2$ (which is consistent with the notion that if the time interval is significant, the difference between the average speed and the instantaneous speed can also be significant).

So suppose we feel that a time interval of 0.001 seconds is sufficiently close to 0; then, thanks to the calculator, it is not difficult to compute the relatively small range of values for the exact speed of the object at the instant $t = 2$. More specifically:

- **Let's first compute the average speed of the object during the time between $t = 1.999$ and $t = 2$.**
  - **-- When $t = 1.999$, $x = 3^{1.999}$ feet $=$ 8.9901179[16] feet**
  - **-- When $t = 2$, $x = 3^2$ feet $=$ 9 feet.**
  - **-- Therefore the distance traveled during the time interval from $t = 1.99$ to $t = 2$ is 9 feet $-$ 8.0001179 feet $=$ 0.0098821 feet**
  - **-- The time it took to travel this distance $=$ 0.001 seconds**
  - **-- Therefore, the average speed during this time interval $=$**

    0.0098821 feet $\div$ 0.001 seconds $=$ 9.8821 feet per second
  - **-- And because the object is accelerating, it means that:**

    $$v_{t=2} > 9.8821 \text{ feet per second}$$

- **Let's next compute the average speed of the object during the time between $t = 2$ and $t = 2.001$**
  - **-- When $t = 2.001$, $x = 3^{2.001}$ feet $=$ 9.0098929 feet**
  - **-- When $t = 2$, $x = 3^2$ feet $=$ 9 feet.**
  - **-- Therefore the distance traveled during the time interval from $t = 2$ to $t = 2.001$ is 9.0098929 feet $-$ 9 feet $=$ 0.0098929 feet**
  - **-- The time it took to travel this distance $=$ 0.001 seconds**
  - **-- Therefore, the average speed during this time interval $=$**

    0.0098929 feet $\div$ 0.001 seconds $=$ 9.8929 feet per second
  - **-- And because the object is accelerating, it means that:**

    $$v_{t=2} < 9.8929 \text{ feet per second}$$

[16] This is a rounded off value. The exact value is a non-terminating, non-repeating decimal.

Thus with just these few calculations we know that the exact speed of the object at the instant $t = 2$ must be greater than 9.8821 feet per second but less than 9.8929 feet per second. So we can say with certainty that to the nearest tenth of a foot per second the exact speed of the object at the instant $t = 2$ is 9.9 feet per second. And if we needed a better approximation for the exact speed at the instant $t = 2$, we would simply repeat the above process but only over an even shorter time interval

7. A Brief Look Ahead:

What we have done so far involves nothing beyond some understanding of arithmetic (basically rates). However, the problem with our approach is that it is labor intensive. For example, our method would be quite cumbersome (not to mention, inconvenient) if we had to go through the calculations every time we wanted to find the speed of the object at different moments in time.

The study of calculus fine tunes the definition of instantaneous speed and also employs algebra to help us find more efficient ways to find the exact speed of an object under any rule of motion and at any time. How this is done is the subject of Part 3 of this presentation.

More specifically, in Part 3 we shall apply the arithmetic-based discussion of this lecture in a more quantitative (algebraic) way to an object moving along the x- axis according to the equation of motion:

$$x = t^3.$$

and our goal will be to find the relationship between the velocity (v) of the object at any time (t).